Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-304 PHYSICAL DYNAMICS

Time Allowed: 2 hours 30 minutes

Date: 10th May, 2011

Time: 14:30 - 17:00

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. Course work comprises 25% of the final mark.**

A formula sheet containing mathematical results that may be of help in various questions is provided at the end of the examination paper.

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr G Travaglini, Dr R Russo

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SECTION A. Attempt answers to all questions.

A1. Consider a single particle of mass m moving in one dimension parameterised by the coordinate x. The particle is subject to a conservative force whose potential is $V(x)$. Write down the Lagrangian of the system, $L(x, \dot{x})$, and give the expression of the momentum conjugate to x. **[4]**

A2. Write down the Lagrange equations for the system described in A1, and use them to show that, if $V(x) = 0$, then the momentum of the particle is a conserved quantity. **[4]**

A3. Again in the case $V(x) = 0$, what is the symmetry that guarantees that momentum is conserved, as a consequence of Noether's theorem? **[3]**

A4. Consider the energy $E(x, \dot{x}) := (1/2)m\dot{x}^2 + V(x)$ of the conservative system in A1. Calculate explicitly the time derivative $\dot{E}(x, \dot{x})$ of the energy, showing that it vanishes upon using the Lagrange equations. **[5]**

A5. Consider the motion of a particle of mass m with one degree of freedom parameterised by a coordinate x in a potential $V(x)$. Explain how the Hamiltonian $H(p, x)$ is obtained from the Lagrangian, and write down its expression. **[4]**

A6. Write down the Hamilton equations for the system described in A5. **[4]**

A7. Define what is meant by the number of degrees of freedom of a mechanical system. **[3]**

A8. Explain what is meant by a set of generalised coordinates for a mechanical system. **[3]**

A9. Consider a system with two degrees of freedom described by the Lagrangian $L = (m/2)(\dot{q}_1^2 + \dot{q}_2^2) - (m/2)\omega^2 q_1^2$, where m and ω are constant quantities. What are the conserved quantities of the system and what are the corresponding Noether symmetries associated to their conservation? **[5]**

A10. Consider a system of two pointlike particles of masses m_1 and m_2, with coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to an inertial frame. Define the position $\mathbf{R}$ of the centre of mass of this system. **[4]**

A11. Explain what is meant by internal forces and external forces acting on the system considered in A10. **[4]**

A12. For a rigid body, define what is meant by a body-fixed frame. **[3]**

A13. Explain what is a principal axis system for a rigid body. **[4]**

SECTION B. Answer two of the four questions in this section.

B1

A conservative mechanical system consists of a pointlike mass M, which is free to slide down a frictionless plane inclined at an angle equal to α, and a pendulum which is suspended from the position of the mass M. The pendulum is made of a massless rod of length l, with a mass m suspended at its other end, as shown in the the figure. The system can move on a fixed inclined plane, and gravity acts as usual along the vertical direction.

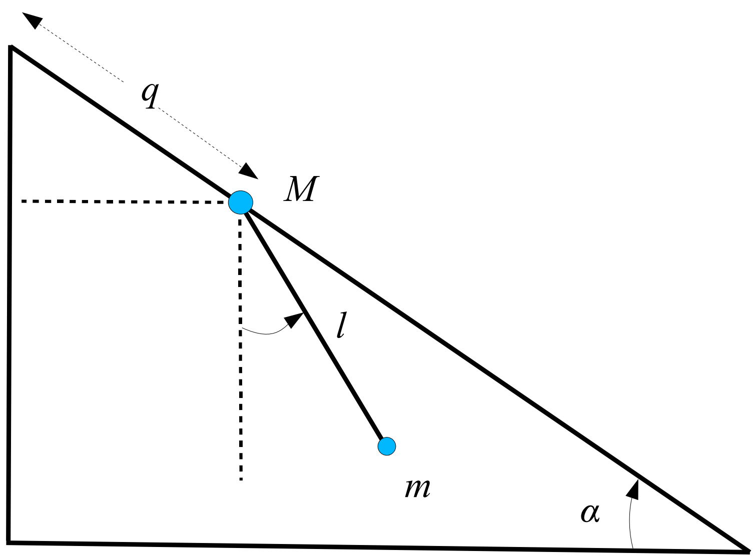
(i) How many degrees of freedom does the system have? **[3]**

(ii) Using the position q of the mass M measured along the inclined plane as one of the generalised coordinates, write down the Lagrangian of the system and the Lagrange equations. **[9]**

(iii) Consider now the case where $\alpha = 0$, i.e. the suspension point of the pendulum moves horizontally. What are the conserved quantities of the system, and what the corresponding Noether symmetries? **[5]**

(iv) Again choosing $\alpha = 0$ as in part (iii), find the equilibrium positions of the system. **[3]**

(v) Find the frequencies of small oscillations about the equilibrium position determined in part (iv). **[5]**



B2

A conservative mechanical system consists of a particle of mass m subject to a central force whose potential is $V(r)$. Here we denote by $\mathbf{r}$ the position of the particle in an inertial reference frame $(\hat{x}, \hat{y}, \hat{z})$ with origin at the centre of the field O, and $r := |\mathbf{r}|$.

(i) Prove that the angular momentum of the particle about the point O is conserved, i.e. prove that $\dot{\mathbf{L}} = 0$. **[4]**

(ii) Orienting the $\hat{z}$ axis of the reference frame along the (constant) direction of the angular momentum, and choosing plane polar coordinates (r, ϕ) in the $(\hat{x}, \hat{y})$ plane, write down the Lagrangian of the system and the Lagrange equations. Furthermore, prove that the momentum p_ϕ associated to ϕ is the angular momentum of the particle. **[5]**

(iii) Write down the expression for the energy of the particle, and use the conservation of p_ϕ to eliminate $\dot{\phi}$ from this expression, thus giving the energy in the form

$$E(r, \dot{r}) \;=\; \frac{1}{2} m \dot{r}^2 + V_{\text{eff}}(r) \; .$$

Write down the expression for the one-dimensional effective potential $V_{\text{eff}}(r)$. **[5]**

(iv) Consider now the case where the potential has the following form,

$$V(r) \;=\; -\frac{v_0}{3r^3} \; ,$$

where v_0 is a positive constant. Determine the radius r_0 of the circular orbit, and the value E_0 of the energy corresponding to this circular orbit. **[5]**

(v) Again in the case where $V(r)$ is equal to the expression given in part (iv), sketch the effective potential and perform a qualitative analysis of the motion of the system as the energy is varied (for fixed angular momentum). For each value of the energy, discuss the qualitative features of the orbit as a function of the initial condition for r. **[6]**

B3

A conservative mechanical system consists of a circular ring of radius a centred at C, mass M and constant mass density, and a pointlike bead of mass m which is free to swing along the ring without friction. The system is suspended from a point O of the ring, and can move in a vertical plane $(\hat{x}, \hat{y})$, so that the ring is free to oscillate about the point O in a vertical plane, while the bead can move along the ring. Gravity acts along the vertical direction (see the figure below for a representation of the system).

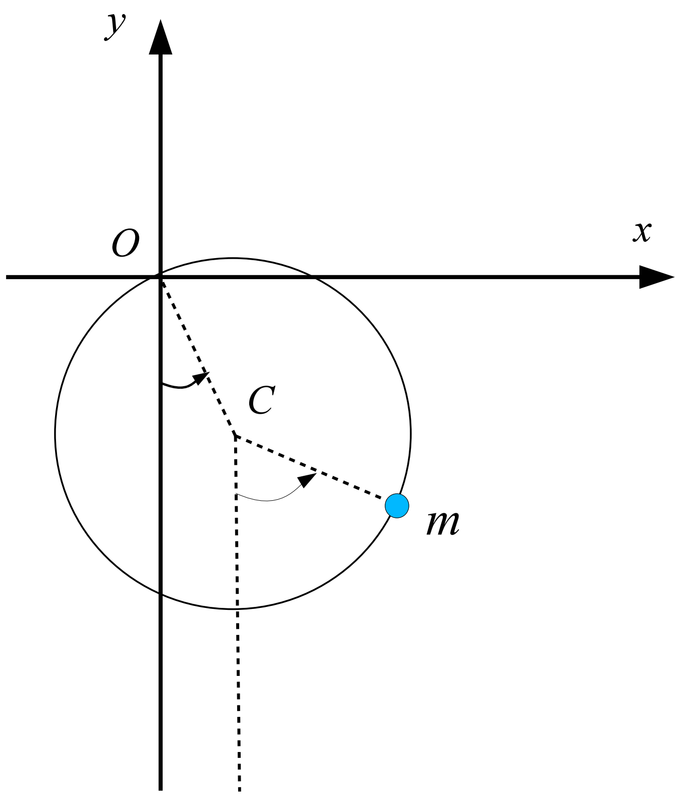
(i) How many degrees of freedom does the system have? **[4]**

(ii) Calculate the moment of inertia of the ring (without the bead) with respect to an axis orthogonal to the plane $(\hat{x}, \hat{y})$ and passing through its centre C. Use the parallel axis theorem to calculate the moment of inertia of the ring with respect to an axis orthogonal to the $(\hat{x}, \hat{y})$ plane and passing through the fixed point O. Assume that the width of the ring is negligible compared to its length. **[5]**

(iii) Write down the Lagrangian of the system ring + bead. **[6]**

(iv) Determine the stable equilibrium position of the system. **[4]**

(v) For the situation where $M = 2\,m$, calculate the frequencies of small oscillations about the stable equilibrium position obtained in part (iv). **[6]**



B4

Consider the following Lagrangian describing the three-dimensional motion of a particle of mass m in an inertial system $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$,

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{m}{2}\omega^2(a\,x_1^2 + b\,x_2^2) ,$$

where a and b are two constant numbers.

(i) Write down the Lagrange equations for the system. **[4]**

(ii) For $a \neq b$, what are the conserved quantities of the system, and what the corresponding Noether symmetries? Consider next the case $a = b$. Discuss if there are any additional symmetries, and find the corresponding conserved quantities. **[5]**

(iii) Write down the Hamiltonian of the system, and the corresponding Hamilton equations for $a \neq b$. **[5]**

(iv) Given a physical observable $\mathcal{O}(\mathbf{x}, \mathbf{p})$, prove that the time evolution of $\mathcal{O}$ is described by the equation

$$\dot{\mathcal{O}} = \{\mathcal{O}, H\} ,$$

where

$$\{A, B\} := \sum_{n=1}^{3} \Big(\frac{\partial A}{\partial x_n}\frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n}\frac{\partial B}{\partial x_n} \Big)$$

is the Poisson bracket of A with B. $\mathbf{x} := (x_1, x_2, x_3)$ and $\mathbf{p} := (p_1, p_2, p_3)$ denote the position vector and the momentum of the particle, respectively. **[6]**

(v) Using the result derived in part (iv), calculate the time evolution of p_3 and L_3, i.e. calculate $\dot{p}_3$ and $\dot{L}_3$, in the case where $a \neq b$. **[5]**

FORMULA SHEET

Plane polar coordinates:

$$
\begin{aligned}
\mathbf{r} &:= r\,\hat{\mathbf{r}}\ , \\
\dot{\mathbf{r}} &= \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\phi}\,\hat{\boldsymbol{\phi}}\ , \\
\dot{\mathbf{r}}^2 &= \dot{r}^2 + r^2\,\dot{\phi}^2\ .
\end{aligned}
$$