SECTION A

A1. $\boxed{L = T - V}$ when $T = \frac{1}{2} m \dot{x}^2$

$$\boxed{p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}}$$

A2. $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$ or $\frac{d}{dt}(m\dot{x}) = -V'(x) \Rightarrow$

$\boxed{m\ddot{x} = -V'(x)}$. If $V(x) = 0$, then

$\frac{d}{dt} p = 0$ with $p \equiv \frac{\partial L}{\partial \dot{x}} \Rightarrow p$ is conserved

A3. Translations along the x-axis

A4. $E(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 + V(x)$

$$\dot{E} = \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial \dot{x}}\ddot{x} = V'(x)\dot{x} + m\dot{x}\ddot{x} =$$

$$= \dot{x}\left(m\ddot{x} + V'(x)\right) = 0 \text{ since } m\ddot{x} = -V'(x)$$

A5. $H = p\dot{x} - L$ with $p \equiv \frac{\partial L}{\partial \dot{x}}(x, \dot{x})$ and

$\dot{x} = \dot{x}(x, p)$ from the equation which defines p.

$$H(p, x) = p \cdot \frac{p}{m} - \frac{1}{2} m \left(\frac{p}{m}\right)^2 + V(x) \Rightarrow$$

$$\boxed{H(p, x) = \frac{p^2}{2m} + V(x)}$$

A6. $$\boxed{\frac{\partial H}{\partial p} = \dot{x} \ , \ \frac{\partial H}{\partial x} = -\dot{p}}$$ or

$$\frac{p}{m} = \dot{x} \ , \quad V'(x) = -\dot{p}$$

A7. The number of degrees of freedom of a mechanical system is the number of independent parameters that one has to specify in order to completely determine the position of the system

A8. A set of generalised coordinates is a set of parameters $\vec{q} = (q_1, \ldots, q_N)$ (where N = number of degrees of freedom) whose values completely specify the position of the system

A9. $$L = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{m}{2}\omega^2 q_1^2$$

There are 2 conserved quantities:

a) the Energy of the system

$$\boxed{E = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{m}{2}\omega^2 q_1^2}$$ and

b) the momentum $p_2 = \frac{\partial L}{\partial \dot{q}_2} = m\dot{q}_2$

The Noether symmetries associated with their conservation are time translations and translations along q_2, respectively

A10. $\boxed{\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}}$ is the definition

of the position $\vec{R}$ of the centre of mass.

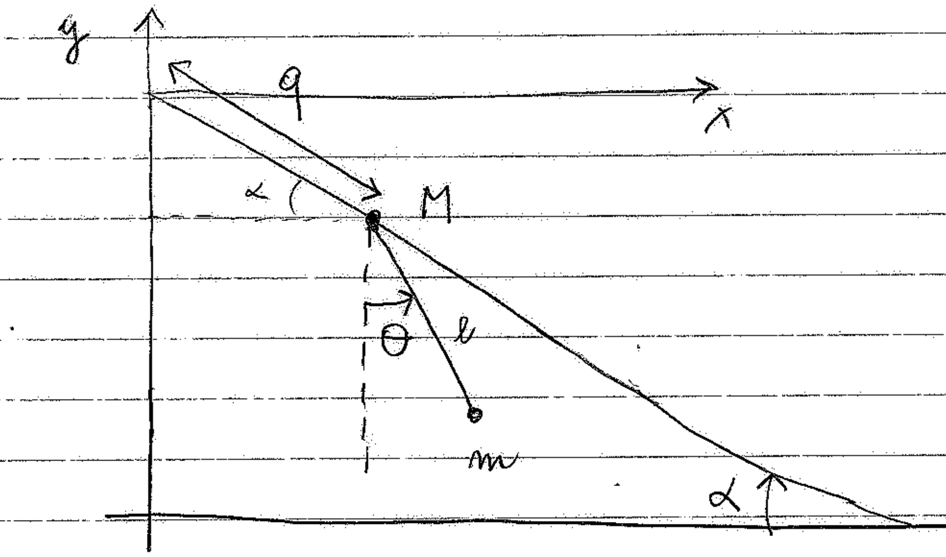
A11. The internal forces are those exerted by particle 1 on particle 2 and by particle 2 on particle 1. The external forces are due to sources outside the system, e.g. gravity.

A12. A <u>body-fixed frame</u> is any reference frame which is fixed with the rigid body and moving with it. In a body-fixed frame, the rigid body is at rest.

A13. A <u>principal axis system</u> is a coordinate system where the inertia tensor is diagonal, i.e. of the form $I = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix}$.

SECTION B

PROBLEM B1



(i) 2 degrees of freedom. We can choose the position q of M measured along the inclined plane, and the angle θ as in the figure.

(ii) Choosing axes as in the figure the positions of M and m are

$$P_M \equiv (q\cos\alpha, -q\sin\alpha)$$

$$P_m = (q\cos\alpha + l\sin\theta, -q\sin\alpha - l\cos\theta)$$

The kinetic energies:

$$T_M = \frac{1}{2} M \dot{q}^2$$

$$T_m = \frac{1}{2} m \left[(\dot{q}\cos\alpha + l\dot{\theta}\cos\theta)^2 + (-\dot{q}\sin\alpha + l\dot{\theta}\sin\theta)^2 \right]$$

$$= \frac{1}{2} m \left[\dot{q}^2 + l^2\dot{\theta}^2 + 2\dot{q}\dot{\theta} l (\cos\theta\cos\alpha - \sin\theta\sin\alpha) \right]$$

=

$$\Rightarrow T_m = \frac{1}{2} m \left[\dot{q}^2 + l^2 \dot{\theta}^2 + 2 l \dot{q} \dot{\theta} \cos(\theta + \alpha) \right]$$

- The potential energy:

$$V = V_M + V_m = M g y_M + m g y_m =$$
$$= g \left[-M q \sin\alpha - m q \sin\alpha - m l \cos\theta \right]$$
$$= -g \left[(M+m) \sin\alpha \, q + m l \cos\theta \right]$$

Hence

$$\boxed{L = T - V = \frac{1}{2} (M+m) \dot{q}^2 + \frac{1}{2} m \left(l^2 \dot{\theta}^2 + 2 l \dot{q} \dot{\theta} \cos(\theta+\alpha) \right) + g \left[(M+m)(\sin\alpha) q + m l \cos\theta \right]}$$

- Euler-Lagrange equations:

- For q: $\frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p_q} = \frac{\partial L}{\partial q}$

$$p_q = \frac{\partial L}{\partial \dot{q}} = (M+m) \dot{q} + m l \dot{\theta} \cos(\theta + \alpha) \Rightarrow$$

$$\boxed{(M+m) \ddot{q} + m l \ddot{\theta} \cos(\theta+\alpha) - m l \dot{\theta}^2 \sin(\theta + \alpha) = = g (M+m) \sin\alpha}$$

- For θ:

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot\theta}}_{p_\theta} = \frac{\partial L}{\partial \theta}$$

$$p_\theta = ml^2\dot\theta \quad \Rightarrow$$

$$\boxed{\cancel{m}l^2\ddot\theta = -\cancel{m}l\ddot q_\theta \sin(\theta+\alpha) - \cancel{m}gl\sin\theta}$$

(iii) If $\alpha=0$, the problem is invariant under translations about the x-axis.

Noether's theorem then predicts that p_q is conserved for $\alpha=0$.

That is indeed the case, as we found in part (ii) that

$$\frac{d}{dt}p_q = g(M+m)\sin\alpha \to 0 \quad \text{as } \alpha=0.$$

(iv) For $\alpha = 0$, the potential is

$\boxed{V = -mgl\cos\theta}$ and obviously

there is a minimum (stable equilibrium position) for $\boxed{\theta = 0}$. Note that x is undetermined (there is invariance under x-translations)

(v) Next we write down the Lagrangian of the small oscillations, $L_{s.o.}$

This is

$$L_{s.o.} = \frac{1}{2}(M+m)\dot{q}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{x}\dot{\theta} - mgl\frac{\theta^2}{2}$$

$$= \frac{1}{2}(\dot{q}\ \ \dot{\theta})\begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}\begin{pmatrix} \dot{q} \\ \dot{\theta} \end{pmatrix} +$$

$$-\frac{1}{2}(q\ \ \theta)\begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} q \\ \theta \end{pmatrix}$$

$$= \frac{1}{2}(\dot{q}\ \dot{\theta})\,T_{s.o.}\begin{pmatrix} \dot{q} \\ \dot{\theta} \end{pmatrix} - \frac{1}{2}(q\ \theta)\,V_{s.o.}\begin{pmatrix} q \\ \theta \end{pmatrix}$$ with

$$T_{s.o.} = \begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}, \quad V_{s.o.} = \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}$$

The frequencies of small oscillations are found by solving the secular equation

$$\boxed{\det\left(V_{s.o.} - \omega^2 T_{s.o.}\right) = 0}$$

or

$$\det\begin{pmatrix} -\omega^2(M+m) & -\omega^2 ml \\ -\omega^2 ml & mgl - m\omega^2 l^2 \end{pmatrix} = 0$$

$$\omega^2(M+m)\cdot ml^2\left(\omega^2 - \frac{g}{l}\right) - (ml)^2\omega^4 = 0$$

- 1^{st} solution : $\omega^2 = 0$ TRIVIAL (associated to translations along the x axis)

Then we have $(M+m)\left(\omega^2 - \frac{g}{l}\right) - m\omega^2 = 0$

$$\omega^2 M = \frac{g}{l}(M+m) \quad \text{or} \quad \boxed{\omega^2 = \frac{g}{l}\left(1 + \frac{m}{M}\right)}$$

PROBLEM B2

(i) $\vec{L} = \vec{r} \times \vec{p}$

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \cancel{\dot{\vec{r}} \times m\dot{\vec{r}}} + \vec{r} \times \vec{F}$$

Since $\vec{F} \parallel \vec{r}$ we conclude $\boxed{\dot{\vec{L}} = \vec{0}}$

or $\vec{L} = \overrightarrow{\text{constant}}$

Alternatively:

the Lagrangian only depends on $|\vec{r}|$, we have invariance under rotations and hence by Noether's theorem, angular momentum is conserved

(ii) $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$

$$\boxed{L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)}$$

Euler - Lagrange equations:

$-\theta$: $$\frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{\theta}}}_{p_\theta} = \frac{\partial L}{\partial \theta}$$

or $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$

- For r : $\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{r}}}_{p_r} = \frac{\partial L}{\partial r}$

$p_r = m\dot{r}$ $\Rightarrow$ $\boxed{m\ddot{r} = mr\dot{\theta}^2 - V'(r)}$

(iii) $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$

using $p_\theta = mr^2\dot{\theta} \equiv l$ and hence

$\dot{\theta} = \frac{l}{mr^2}$ we get

$$E = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{l^2}{2mr^2} + V(r)}_{V_{eff}}$$ or

$\boxed{E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)}$ with $\boxed{V_{eff}(r) = \frac{l^2}{2mr^2} + V(r)}$

This is a one-dimensional problem with potential $V_{eff}(r)$

(IV) For $V(r) = -\frac{v_0}{3r^3}$;

$$V_{eff}(r) = \frac{\ell^2}{2mr^2} - \frac{v_0}{3r^3}$$

The circular orbit occurs for r_0 such that $\boxed{V'(r_0) = 0}$ $\Rightarrow$

$$-\frac{\ell^2}{mr_0^3} + \frac{v_0}{r_0^4} = 0 \Rightarrow \boxed{r_0 = \frac{mv_0}{\ell^2}}$$

The corresponding value of the energy:

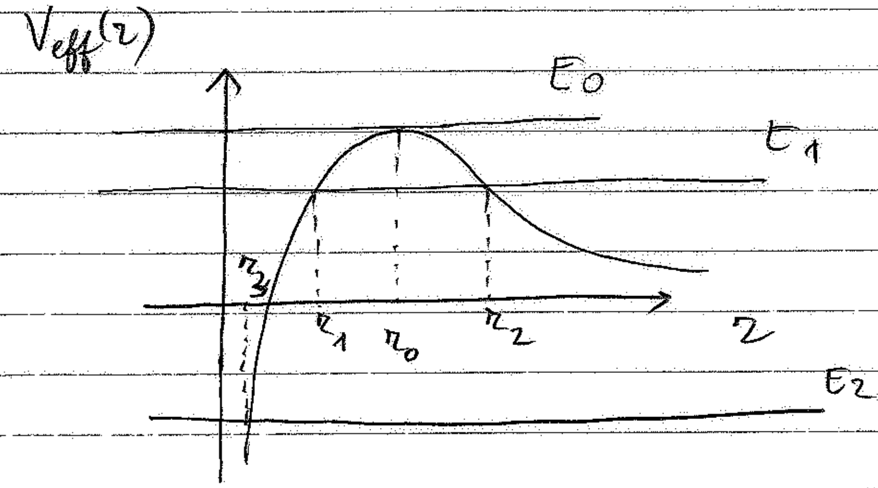
at r_0 we have $\dot{r} = 0$ and

$$E_0 = V(r_0) = \frac{\ell^2}{2mr_0^2} - \frac{v_0}{3r_0^3} =$$

$$= \frac{\ell^2}{2m}\frac{\ell^4}{m^2v_0^2} - \frac{v_0}{3}\frac{\ell^6}{m^3v_0^3} = \left(\frac{1}{2} - \frac{1}{3}\right)\frac{\ell^6}{m^3v_0^2}$$

$$\boxed{E(r_0) = \frac{1}{6}\frac{\ell^6}{m^3v_0^2}}$$

(v) To perform a qualitative study we first plot $V_{eff}(r)$:



$V_{eff}(r) = \frac{\ell^2}{2mr^2} - \frac{\alpha_0}{3r^3}$

* the value $E = E_0$ has already been considered and leads to a circular orbit of radius r_0

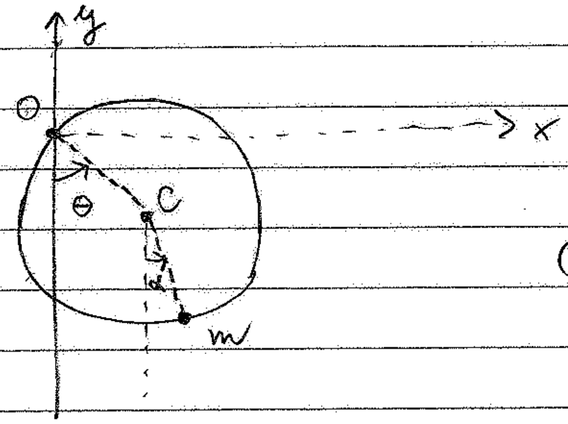
Two more basic situations:

* For $E = E_1$, the motion occurs either in $(0, r_1)$ (and hence is limited), or for $r > r_2$ (and then is not limited).
Which of the two situations occurs is determined by the initial value for r.

* For $E = E_2$ we can only have $r \in (0, r_3) \Rightarrow$ limited motion.

RMK: the qualitative analysis above cannot say whether the limited motions are also closed.

PROBLEM B3



(i) The system has 2 d.o.f which we can parametrise using the angles θ, α as in the figure

(ii) The moment of inertia of the ring with respect to an axis passing through the centre of the ring and perpendicular to it is

$$I_C^{RING} = \int dm\, a^2 = \int ds\, \rho \cdot a^2 = \quad \text{since } \rho = \frac{M}{2\pi a}$$

$$= \frac{M a^2}{2\pi a} \int_0^{2\pi} d\varphi\, a = M a^2 \Rightarrow \boxed{I_C^{RING} = M a^2}$$

In order to calculate the moment of inertia of the ring with respect to an axis orthogonal to the ring and passing through the suspension point O we apply the parallel axis theorem:

$$I_O^{RING} = I_C^{RING} + M\,\overline{OC}^2 = M a^2 + M a^2 \Rightarrow$$

$$\boxed{I_O^{RING} = 2 M a^2}$$

(iii) $L = T - V$

$$T = T_{RING} + T_m$$

$$T_{RING} = \frac{1}{2} I_0^{RING} \dot{\theta}^2 = M a^2 \dot{\theta}^2$$

To calculate T_m, we write first the coordinates of the point m:

$$P_m \equiv (a \sin\theta + a \sin\alpha \, , \, - a\cos\theta - a\cos\alpha)$$

$$T_m = \frac{1}{2} m (\dot{x}_m^2 + \dot{y}_m^2) =$$

$$= \frac{1}{2} m a^2 \left[(\dot{\theta}\cos\theta + \dot{\alpha}\cos\alpha)^2 + (\dot{\theta}\sin\theta + \dot{\alpha}\sin\alpha)^2 \right] =$$

$$= \frac{1}{2} m a^2 \left[\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\theta}\dot{\alpha} (\cos\theta\cos\alpha + \sin\theta\sin\alpha) \right] =$$

$$= \frac{1}{2} m a^2 \left[\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\theta}\dot{\alpha}\cos(\theta - \alpha) \right]$$

$$\Rightarrow \quad T = \frac{1}{2} m a^2 \left[\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\theta}\dot{\alpha}\cos(\theta-\alpha) \right] + M a^2 \dot{\theta}^2$$

$$V = V_{RING} + V_m = - M g a \cos\theta - m g a (\cos\theta + \cos\alpha)$$

Hence

$$\boxed{L = \frac{1}{2} m a^2 \left[\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\theta}\dot{\alpha}\cos(\theta-\alpha) \right] + M a^2 \dot{\theta}^2 + a g \{ (M+m)\cos\theta + m\cos\alpha \}}$$

The Euler-Lagrange equations [NOT ASKED]

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = ma^2\left[\dot\theta + \dot\alpha\cos(\theta-\alpha)\right] + 2Ma^2\dot\theta$$

$$p_\alpha = \frac{\partial L}{\partial \dot\alpha} = ma^2\dot\alpha + ma^2\dot\theta\cos(\theta-\alpha)$$

* $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \theta} \Rightarrow$

$$ma^2\left[\ddot\theta + \ddot\alpha\cos(\theta-\alpha) - \dot\alpha(\cancel{\dot\theta}-\dot\alpha)\sin(\theta-\alpha)\right] + 2Ma^2\ddot\theta =$$

$$= -\cancel{ma^2\dot\theta\dot\alpha}\sin(\theta-\alpha) - (M+m)ag\sin\theta$$

$$\Rightarrow \boxed{ma^2\left[\ddot\theta + \ddot\alpha\cos(\theta-\alpha) + \dot\alpha^2\sin(\theta-\alpha)\right] + 2Ma^2\ddot\theta = -(M+m)a^2\ddot\theta}$$

* $\frac{d}{dt}\frac{\partial L}{\partial \dot\alpha} = \frac{\partial L}{\partial \alpha} \Rightarrow$

$$ma^2\left[\ddot\alpha + \ddot\theta\cos(\theta-\alpha) - \dot\theta(\dot\theta-\cancel{\dot\alpha})\sin(\theta-\alpha)\right] =$$

$$= +\cancel{ma^2\dot\theta\dot\alpha}\sin(\theta-\alpha) - mag\sin\alpha \quad \text{or}$$

$$\boxed{ma^2\left[\ddot\alpha + \ddot\theta\cos(\theta-\alpha) - \dot\theta^2\sin(\theta-\alpha)\right] = -mag\sin\alpha}$$

(iv) The equilibrium positions:

$$\boxed{V(\theta,\alpha) = -ag\{(M+m)\cos\theta + m\cos\alpha\}}$$

$$\frac{\partial V}{\partial \theta} = ag(M+m)\sin\theta \qquad \frac{\partial V}{\partial \alpha} = agm\sin\alpha$$

Obviously the stable equilibrium position occurs for $\boxed{\theta = \alpha = 0}$

Indeed we can also calculate the hessian:

$$\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta=\alpha=0} = ag(M+m)\cos\theta\Big|_{\theta=0} = ag(M+m)$$

$$\left.\frac{\partial^2 V}{\partial \alpha^2}\right|_{\theta=\alpha=0} = agm$$

$$\left.\frac{\partial^2 V}{\partial \alpha \partial \theta}\right|_{\theta=\alpha=0} = 0 \quad \Rightarrow$$

$$V^{(2)} = \begin{pmatrix} ag(M+m) & 0 \\ 0 & agm \end{pmatrix} \quad \text{and}$$

$\det V^{(2)} > 0 \Rightarrow$ minimum

[the discussion of the hessian is not required].

(v) Small oscillations:

the Lagrangian of the small oscillators:

$$T_{s.o.} = \frac{1}{2} m a^2 \left[\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\theta}\dot{\alpha} \right] + M a^2 \dot{\theta}^2$$

$$V_{s.o} = \frac{1}{2} a g \left\{ \theta^2 (M+m) + \alpha^2 m \right\}$$

or $$T_{s.o.} = \frac{1}{2} (\dot{\theta} \;\; \dot{\alpha}) \begin{pmatrix} (m+2M)a^2 & ma^2 \\ ma^2 & ma^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\alpha} \end{pmatrix}$$

$$V_{s.o.} = \frac{1}{2} (\theta \cdot \alpha) \begin{pmatrix} (M+m)ag & 0 \\ 0 & mag \end{pmatrix} \begin{pmatrix} \theta \\ \alpha \end{pmatrix}$$

The secular equation determines the frequencies of small oscillations:

$$\boxed{\det \left(V^{(2)} - \omega^2 T^{(2)} \right) = 0}$$ where

$$T^{(2)} = a^2 \begin{pmatrix} m+2M & m \\ m & m \end{pmatrix} \qquad V^{(2)} = ag \begin{pmatrix} M+m & 0 \\ 0 & m \end{pmatrix}$$

Hence we need to calculate

$$\det \begin{pmatrix} ag(M+m) - \omega^2 a^2 (m+2M) & -\omega^2 a^2 m \\ -\omega^2 a^2 m & agm - \omega^2 a^2 m \end{pmatrix} =$$

$$= \left[ag(M+m) - \omega^2 a^2 (m+2M)\right]\left[agm - \omega^2 a^2 m\right] - (\omega^2 a^2 m)^2 =$$

$$= \omega^4 a^4 \left[m(\cancel{m}+2M) - \cancel{m^2}\right] - \omega^2 a^3 g \left[m(m+2M) + m(M+m)\right] +$$

$$+ (ag)^2 m(M+m) = 0 \quad \text{or}$$

$$\omega^4 a^2 \cdot (2M) - \omega^2 ga\left[3M + 2m\right] + g^2 (M+m) = 0 \quad \text{or}$$

$$\omega^4 - \omega^2 \left(\frac{g}{a}\right)\left(\frac{3}{2} + \frac{m}{M}\right) + \frac{g^2}{2a^2}\left(1 + \frac{m}{M}\right) = 0.$$

The solutions are $\boxed{\omega_{(1)}^2 = \frac{g}{2a}, \quad \omega_{(2)}^2 = \frac{g}{a}\left(1 + \frac{m}{M}\right)}$

In order to simplify the calculation I asked to consider only the case where $M = 2m$. In this case

$$\boxed{\omega_{(1)}^2 = \frac{g}{2a} \quad \text{and} \quad \omega_{(2)}^2 = \frac{3g}{2a}}$$

PROBLEM B4

(i) $$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right) - \frac{1}{2} m\omega^2 \left(a x_1^2 + b x_2^2 \right)$$

Euler-Lagrange equations:

* x_1 : $p_1 = m\dot{x}_1$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1} \Rightarrow \cancel{m}\ddot{x}_1 = -\cancel{m} a \omega^2 x_1$$

* x_2 : similarly $\cancel{m}\ddot{x}_2 = -\cancel{m} b \omega^2 x_2$

* x_3 : $\ddot{x}_3 = 0$

(ii) (A) L does not depend explicitly on time hence the energy E is conserved, with

$$E = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right) + \frac{1}{2} m\omega^2 \left(a x_1^2 + b x_2^2 \right)$$

(B) L does not depend on x_3 hence p_{x_3} is conserved where $p_{x_3} = m\dot{x}_3$

(C) For $a = b$, the potential becomes

$V = \frac{1}{2} m \omega^2 (x_1^2 + x_2^2)$, which is invariant under rotations in the $(\hat{x}_1, \hat{x}_2)$ plane

$\Rightarrow L_3$ is conserved with

$$L_3 = x_1 p_2 - x_2 p_1$$

(iii) The Hamiltonian is

$$H(\vec{p}, \vec{x}) = \vec{p} \cdot \dot{\vec{x}} - L = p_1 m\dot{x}_1 + p_2 m\dot{x}_2 + p_3 m\dot{x}_3 +$$

$$- \frac{1}{2} m (p_1^2 + p_2^2 + p_3^2) \frac{1}{m^2} + \frac{1}{2} m\omega^2 (a x_1^2 + b x_2^2) \Rightarrow$$

$$\boxed{H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2} m\omega^2 (a x_1^2 + b x_2^2)}$$

the Hamilton equations: $\frac{\partial H}{\partial \vec{p}} = \dot{\vec{x}}$, $\frac{\partial H}{\partial \vec{x}} = -\dot{\vec{p}}$

$$\Rightarrow \boxed{\begin{array}{llll} \dot{x}_1 = \frac{p_1}{m} & \dot{x}_2 = \frac{p_2}{m} & \dot{x}_3 = \frac{p_3}{m} & \text{and} \\ \dot{p}_1 = -m\omega^2 a x_1 & \dot{p}_2 = -m\omega^2 b x_2 & \dot{p}_3 = 0. & \end{array}}$$

(iv) If $O = O(\vec{x}, \vec{p})$ then

$$\dot{O} = \frac{\partial O}{\partial \vec{x}} \cdot \dot{\vec{x}} + \frac{\partial O}{\partial \vec{p}} \cdot \dot{\vec{p}} = \text{using Hamilton's equations,}$$

$$= \frac{\partial O}{\partial \vec{x}} \cdot \frac{\partial H}{\partial \vec{p}} - \frac{\partial O}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{x}} = \{O, H\}$$

$$\Rightarrow \boxed{\dot{O} = \{O, H\}}$$

(v) We have $\dot{p}_3 = \{p_3, H\}$ and

$\dot{L}_3 = \{L_3, H\}$

- $\dot{p}_3 = \{p_3, H\} = -\frac{\partial p_3}{\partial p_3} \frac{\partial H}{\partial x_3} = 0$ since H is x_3-independent.

- $\dot{L}_3 = \{x_1 p_2 - x_2 p_1, H\} =$

$$= x_1 \{p_2, H\} + \{x_1, H\} p_2 - x_2 \{p_1, H\} - \{x_2, H\} p_1 =$$

$$= x_1 \left(-\frac{\partial p_2}{\partial p_2} \frac{\partial H}{\partial x_2} \right) + \left(\frac{\partial x_1}{\partial x_1} \frac{\partial H}{\partial p_1} \right) p_2 +$$

$$- x_2 \left(- \frac{\partial p_1}{\partial p_1} \frac{\partial H}{\partial x_1} \right) - \left(\frac{\partial x_2}{\partial x_2} \frac{\partial H}{\partial p_2} \right) p_1 =$$

$$= - x_1 \cdot (m\omega^2 b) x_2 + \cancel{\frac{p_1 p_2}{m}} + (m\omega^2 a) x_1 x_2 - \cancel{\frac{p_1 p_2}{m}}$$

$$= m\omega^2 x_1 x_2 (a - b) \Rightarrow$$

$$\boxed{\dot{L}_3 = m\omega^2 x_1 x_2 (a - b)}$$