HOW TO DO NORMAL MODE PROBLEMS

For any conservative mechanical system with n degrees of freedom the Lagrangian L has the general form

$$L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n) = T - V = \frac{1}{2}\sum_{i,j=1}^{n} m_{ij}(q_1,q_2,\ldots,q_n)\,\dot{q}_i\,\dot{q}_j - V(q_1,q_2,\ldots,q_n) \quad .$$

An equilibrium point for the system is a point $(q_1^0,q_2^0,\ldots,q_n^0)$ at which all the generalised forces vanish, $Q_j = -\partial V/\partial q_j = 0$, that is to say, it is an extremum of the potential energy. We can expand the potential energy function in Taylor's series about this point up through terms of second order in the displacements from equilibrium, $q_j - q_j^0$.

$$V(q_1,\ldots,q_n) = V(q_1^0,\ldots,q_n^0) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_{q=q^0}(q_i - q_i^0)(q_j - q_j^0) + \cdots \quad .$$

The small oscillation approximation results if we evaluate all the coefficient functions m_{ij} at the equilibrium point q^0, $m_{ij}^0 = m_{ij}(q_1^0,\ldots,q_n^0)$, and neglect all higher order terms in the Taylor's expansion of the potential energy. If we then use as generalised coordinates the *displacements* from equilibrium, $\eta_i = q_i - q_i^0$, the Lagrangian reduces to the form

$$L = \frac{1}{2}\sum_{i,j=1}^{n} m_{ij}^0 \dot{\eta}_i\dot{\eta}_j - \frac{1}{2}\sum_{i,j=1}^{n} V_{ij}\eta_i\eta_j \quad ,$$

where

$$V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_{q=q^0}$$

The Lagrange equations of motion then take the form

$$\sum_{j=1}^{n}(m_{ij}^0 \frac{d^2\eta_j}{dt^2} + V_{ij}\eta_j) = 0 \quad , i = 1,2,\ldots,n$$

Assume a solution of the form $\eta_j = a_j e^{i\omega t}$ to get a set of linear equations for the amplitudes a_j and the frequency ω,

$$\sum_{j=1}^{n}(V_{ij} - \omega^2 m_{ij}^0)a_j = 0 \quad , i = 1,2,\ldots,n$$

These are a form of eigenvalue problem and have a non-trivial solution only if the secular equation is satisfied,

$$Det\,(V_{ij} - \omega^2 m_{ij}^0) = 0 \quad .$$

This equation has n solutions for ω which are called normal mode frequencies or eigenfrequencies. For each normal mode frequency ω_p, we can substitute into the amplitude equations to find the corresponding a_j^p. Often we can use symmetry arguments to "guess" the form of the amplitudes.

If any root ω^2 is negative, then the equilibrium point is an unstable one. If any root is zero, this corresponds to a rigid translation of the system with no oscillation. It may happen that we have a multiple root. If this happens there will be different amplitude solutions for each of the equal roots and we call this a degeneracy (just as in quantum mechanics). The most general solution for the system is a superposition of these normal mode solutions. Note that for a system of masses held together by ideal springs or elastic bands, the potential energy is exactly quadratic from the beginning. The effect of this is that the second derivatives which define the potential matrix V are constants which are independent of the equilibrium positions q_j^0. Thus for such special systems we can find V_{ij} without finding the values of the equilibrium positions. In the general case however, V_{ij} does depend on the value of the q_j^0 so we must first find the equilibrium point before we can solve the small oscillation equations.