QUESTION 1: Find the energy eigenfunctions $\psi_E(x)$ for a free particle of mass m moving on a circle of circumference L. It is useful to think of the circle as the real line x with periodic identification of points $x \sim x + L$; what boundary condition does this imply for the wavefunction? Hence, what are the allowed values of E? Comment on the limit $L \to \infty$. How does the spacing between energy eigenvalues depend on the size of the circle L?

QUESTION 2: Which of the following 1D (wave) functions are normalisable?

$\psi_1 = e^{-x^2}$, $\psi_2 = e^{-ipx}$ with $i^2 = -1$ and p real, $\psi_3 = |x|^{-1/4}$, $\psi_4 = 1/(1+x)$.
Sometimes you need to plot the wavefunction to see if it diverges for a finite values of x and investigate the integral more closely there.
Comment: ψ_3 is an example of a function that does not vanish rapidly enough as $x \to \pm\infty$, so vanishing of the wave function as $x \to \pm\infty$ is a necessary but not sufficient condition. Assuming that the wave function is approximately $\psi \sim x^{-a}$ as $x \to +\infty$ find a lower bound for a such that ψ is normalisable at $x \to \infty$.

QUESTION 3: Which of the following operators are *linear*?

$\hat{A}\Psi(x) = [\Psi(x)]^2$, $\hat{B}\Psi(x) = x^2 \frac{d}{dx}\Psi(x)$, $\hat{C}\Psi(x) = \sin(\Psi(x))$, $\hat{D}\Psi(x) = \int_{x'=0}^{x'=x} dx' \Psi(x')$.

QUESTION 4: A particle is trapped in the escape-proof box (see lecture notes set 1) with potential $V = 0$ for $-L/2 \leq x \leq L/2$ and $V = \infty$ otherwise. The normalised energy eigenstates are $\psi_n(x) = \sqrt{\frac{2}{L}}\cos(n\pi x/L)$ for $n = 1, 3, 5, \ldots$ and $\psi_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)$ for $n = 2, 4, 6, \ldots$. Let $\langle A \rangle_n$ be the expectation value of any operator $\hat{A}$ in the state $\psi_n(x)$. Find $\langle x \rangle_n$ using symmetry considerations. Show that

$$\langle (x - \langle x \rangle_n)^2 \rangle_n = \frac{L^2}{12}(1 - \frac{6}{n^2\pi^2}) \ .$$

Restrict yourself to the case of odd n odd and you may use the formula

$$\int_{-b}^{b} \cos(ax)^2 x^2 dx = \frac{4a^3b^3 + \left(6a^2b^2 - 3\right)\sin(2ab) + 6ab\cos(2ab)}{12a^3} \ .$$

Furthermore, show that the classical result is recovered in the limit of large quantum number $n \to \infty$. (Hint: a classical particle bounces back and forth and is equally likely to be anywhere in the box).

QUESTION 5: *Commutator Yoga.* Recall the definition of the operator for position $\hat{x} = x$ and momentum $\hat{p} = -i\hbar\partial/\partial x$ in one dimension. Calculate the following commutators:

1. $[x^2, \hat{p}]$
2. $[x^2, \hat{p}^2]$
3. $[\hat{T}, x\hat{p}]$ with $\hat{T} = \hat{p}^2/(2m)$
4. $[V(x), x\hat{p}]$ with $V(x) = kx^n$

You may calculate these commutators by directly acting with the operators on an arbitrary wavefunction $\psi(x)$ (yawn) or you use the fundamental commutation relation $[x, \hat{p}] = i\hbar$ and the Leibniz rules for operators $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$ and $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$ (nifty method).

QUESTION 6: A particle of mass m moves in one dimension subject to the potential $V = \frac{1}{2}m\omega^2x^2$ (simple harmonic oscillator). Express the expectation value of the energy in terms of $\langle x \rangle$, $\langle p \rangle$, Δx and Δp. Using the uncertainty relation for momentum and position show that $\langle E \rangle \geq \hbar\omega/2$.