CONDUCTORS AND MAGNETIC FIELDS

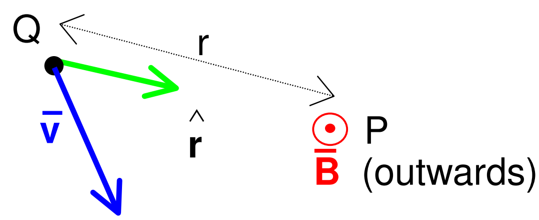
This handout covers:

- The Biot-Savart Law for $d\overline{\mathbf{B}}$ due to a current element
- Force on a current-carrying wire in a magnetic field
- Torque on a current loop and the magnetic dipole
- Ampere's Law

Magnetic field due to a current-carrying wire: the Biot–Savart Law

Recall: $\overline{\mathbf{B}}$ at a point P due to a point charge, Q, moving with velocity **v**, is given by

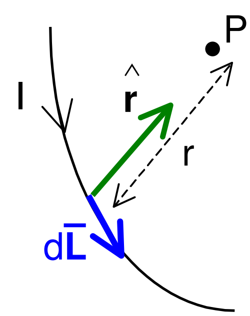
$$\overline{\mathbf{B}} = \frac{\mu_o Q}{4\pi r^2}\left[\overline{\mathbf{v}} \times \hat{\mathbf{r}}\right]$$



Let a thin wire carry a current I.

Let a short length $d\overline{\mathbf{L}}$ contain charge dQ moving with velocity $\overline{\mathbf{v}}$

The contribution of $d\overline{\mathbf{L}}$ to the magnetic field at P is



$$d\overline{\mathbf{B}} = \frac{\mu_o dQ}{4\pi r^2}\left[\overline{\mathbf{v}} \times \hat{\mathbf{r}}\right]$$

But $I = \dfrac{dQ}{dt}$ and $\mathbf{v} = \dfrac{d\mathbf{L}}{dt}$ $\Rightarrow$ $d\overline{\mathbf{B}} = \dfrac{\mu_o I}{4\pi r^2}\left[d\overline{\mathbf{L}} \times \hat{\mathbf{r}}\right]$

The Biot-Savart Law

To find the total field at P, we must integrate this expression along the whole length of the wire.

Special case: Magnetic field due to a straight conductor

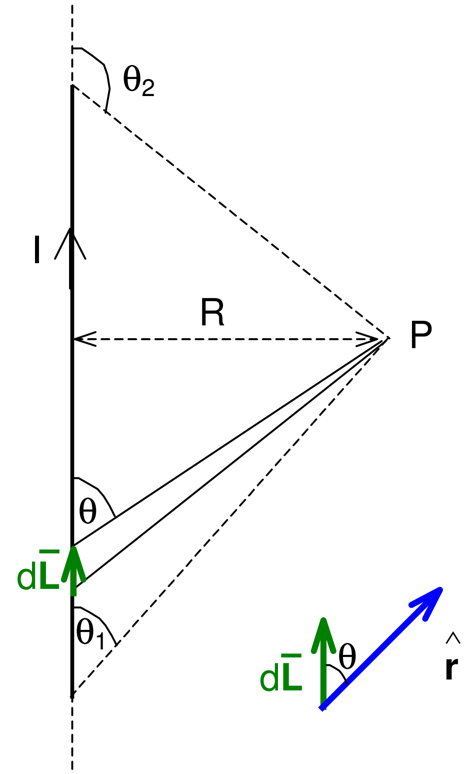
What is $\overline{\mathbf{B}}$ at a point P which is at a perpendicular distance R from a straight wire?

Consider a small element $d\overline{\mathbf{L}}$

θ = angle between the wire and the line joining $d\overline{\mathbf{L}}$ and P

r = distance from $d\overline{\mathbf{L}}$ to P

Its contribution to $\overline{\mathbf{B}}$ at P is

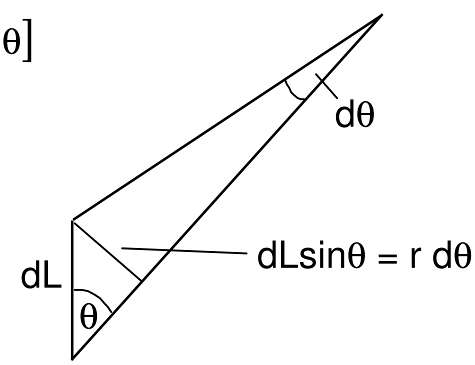


$$d\overline{\mathbf{B}} = \frac{\mu_o I}{4\pi r^2}\left[d\overline{\mathbf{L}} \times \hat{\mathbf{r}}\right]$$

$$\left| d\overline{\mathbf{L}} \times \hat{\mathbf{r}} \right| = dL\sin\theta$$

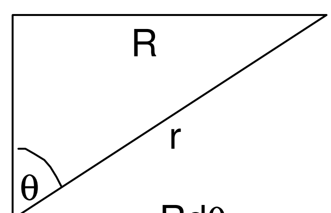
From the right hand rule, the direction of $d\overline{\mathbf{B}}$ is into the page.

The magnitude of dB is $$dB = \frac{\mu_o I}{4\pi r^2}\left[dL\sin\theta\right]$$



But $R = r\sin\theta$

and $dL\sin\theta = r d\theta$



So $r = \frac{R}{\sin\theta}$ and $dL = \frac{R d\theta}{\sin^2\theta}$

Therefore $$dB = \frac{\mu_o I}{4\pi}\frac{\sin^2\theta}{R^2}\frac{R d\theta}{\sin^2\theta}\sin\theta$$

$$\Rightarrow \quad dB = \frac{\mu_o I \sin\theta}{4\pi R}$$ **Contribution of element dL to the field at P**

To find the total field, we INTEGRATE from one end of the wire to the other – i.e., from θ_1 to θ_2.

Example: An infinitely long wire: $\theta_1 = 0$ and $\theta_2 = 180^o$.

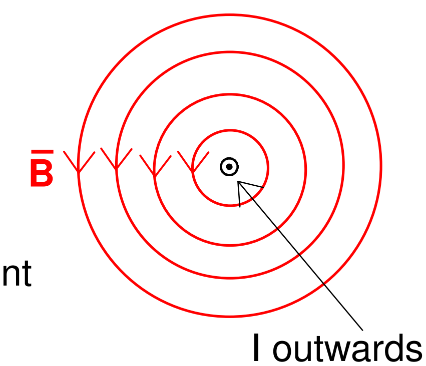
$$dB = \frac{\mu_o I}{4\pi R}\int_0^{\pi} \sin\theta d\theta = \frac{\mu_o I}{4\pi R}\left[-\cos\theta\right]_0^{\pi} = \frac{\mu_o I}{4\pi R}\left[-1 - 1\right]$$

$$\Rightarrow \quad B = \frac{\mu_o I}{2\pi R}.$$

Magnetic field pattern of a straight current-carrying wire:

As we have seen before, the field lines are circular loops.

Are the lines of $\overline{\mathbf{B}}$ clockwise or anticlockwise?



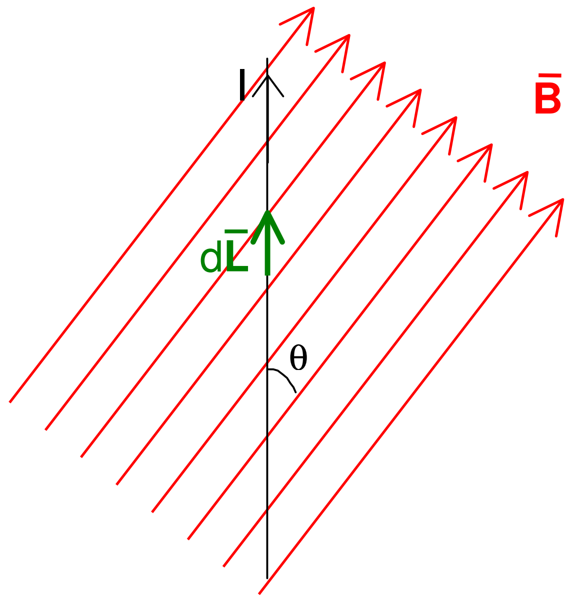
To decide, we can use another form of the right hand rule:

- Let the thumb point in the direction of the current
- The fingers will then curl in the direction of $\overline{\mathbf{B}}$.

The force on a current-carrying wire in a magnetic field

In a current carrying wire, charges are moving. Therefore, if the wire is in a magnetic field, a force will be exerted on the moving charges and hence on the wire. This is the principle of the electric motor.

Consider a section of straight wire carrying current I in a magnetic field $\overline{\mathbf{B}}$. Let $d\overline{\mathbf{L}}$ contain charge dQ moving with velocity **v**.



The force on $d\overline{\mathbf{L}}$ is

$$d\overline{\mathbf{F}} = dQ(\overline{\mathbf{v}} \times \overline{\mathbf{B}})$$

But $I = \dfrac{dQ}{dt}$ and $\overline{\mathbf{v}} = \dfrac{d\overline{\mathbf{L}}}{dt}$

Therefore $d\overline{\mathbf{F}} = I(d\overline{\mathbf{L}} \times \overline{\mathbf{B}})$ THE MOTOR EQUATION

The magnitude of $d\overline{\mathbf{F}}$ is $dF = BIdL\sin\theta$

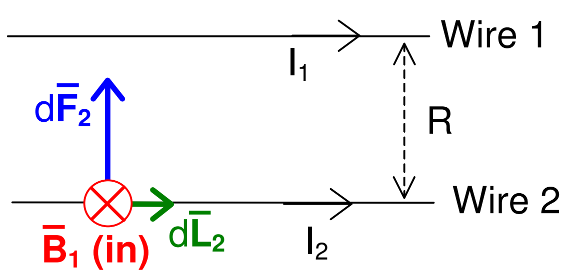
Force between two parallel wires

Consider two parallel wires, separation R, carrying currents I_1 and I_2.

Let $\overline{\mathbf{B}}_1$ = field at wire 2 due to wire 1

$\overline{\mathbf{B}}_2$ = field at wire 1 due to wire 2

Consider an element $d\overline{\mathbf{L}}$ in wire 2



By the right hand rule: $\overline{\mathbf{B}}_1$ is into the paper at wire 2

Force on dL_2, $d\overline{\mathbf{F}}_2$ is $$d\overline{\mathbf{F}}_2 = I(d\overline{\mathbf{L}}_2 \times \overline{\mathbf{B}}_1)$$

Direction: Towards wire 1
Magnitude: $dF_2 = B_1 I_2 dL_2$

Recall, at perpendicular distance R from a straight wire, $B = \dfrac{\mu_o I}{2\pi R}$

Therefore, $$dF_2 = \frac{\mu_o I_1 I_2 dL_2}{2\pi R}.$$

$\Rightarrow$ Force per unit length is $$\frac{dF_2}{dL_2} = \frac{\mu_o I_1 I_2}{2\pi R}$$

If $I_1 = I_2 = I$, then $$\frac{dF_2}{dL_2} = (2 \times 10^{-7})\frac{I^2}{R} \quad \text{N m}^{-1}$$

SI definition of the Ampere:

1 Ampere is the current which, when flowing in two long straight parallel wires separated by 1 m, produces a force per unit length between them of 2×10^{-7} N m^{-1}.

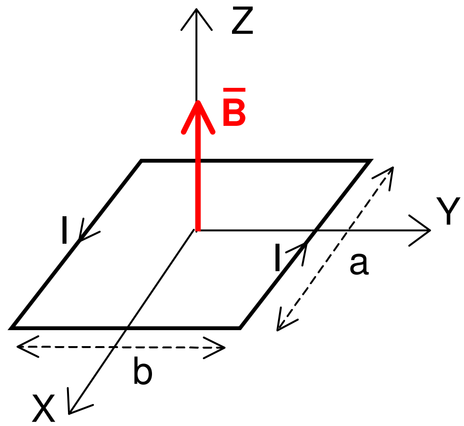
Torque on a current loop: The magnetic dipole

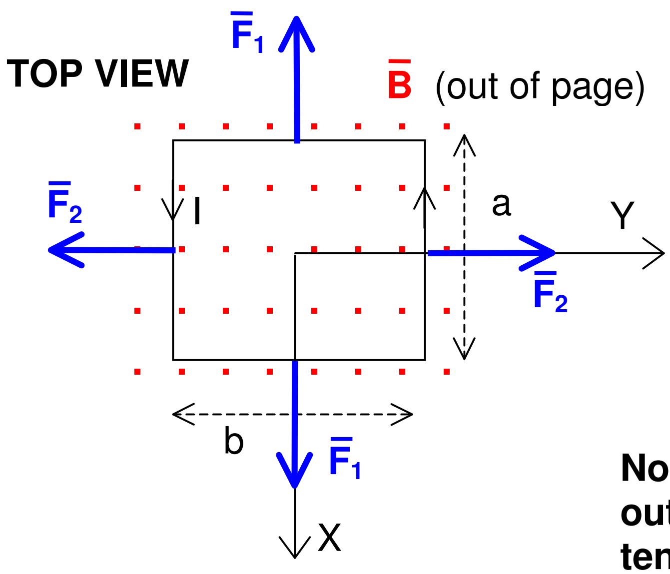
Consider a rectangular loop carrying current I in a uniform magnetic field $\overline{\mathbf{B}}$.

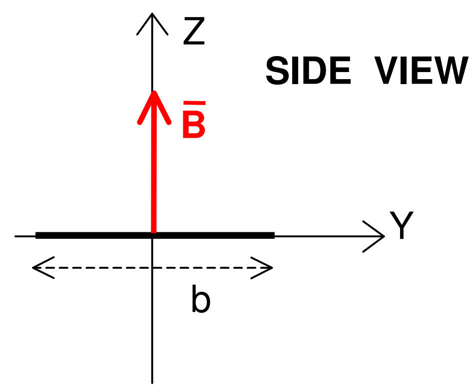
(i) Plane of the loop perpendicular to **B**:

Let the loop be in the X-Y plane with $\overline{\mathbf{B}}$ in the Z direction.

Note that the current is always perpendicular to $\overline{\mathbf{B}}$.

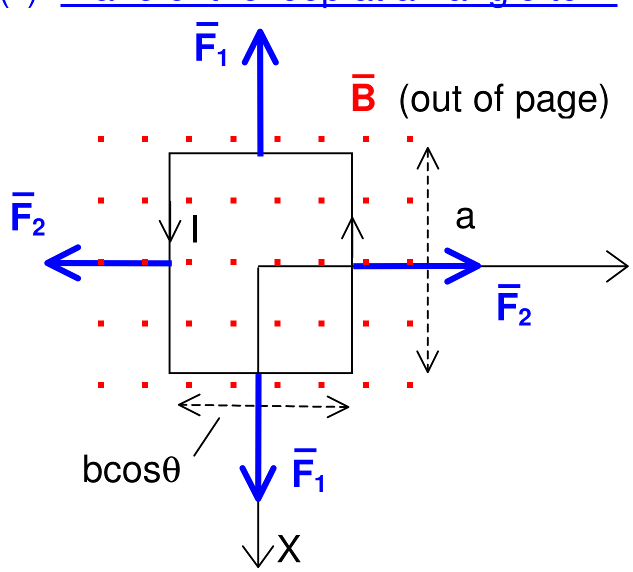


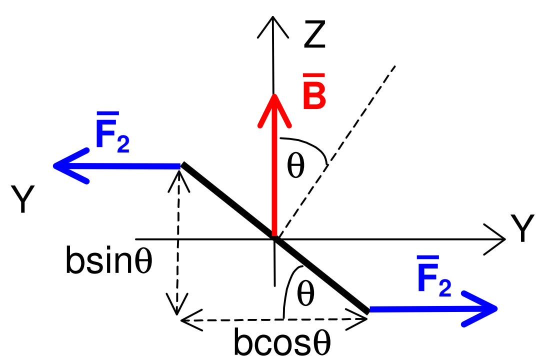




Note how all the forces cancel out in this situation – they just tend to stretch the loop.

(ii) Plane of the loop at an angle to **B**:





$F_1 = BI(b\cos\theta)$ $F_2 = BIa$ (sides a are still perpendicular to $\overline{\mathbf{B}}$)

Forces $\overline{\mathbf{F}}_1$ cancel out. But **forces $\overline{\mathbf{F}}_2$ don't** cancel. Although they are still equal and opposite, they no longer act along the same line, so they form a TORQUE.

Torque = (Force)(Perpendicular distance)

$\tau = (BIa)(b\sin\theta) = \mu B\sin\theta$ where $\mu = Iab$

This looks like the magnitude of a vector cross product, which is exactly what it is:

$$\overline{\tau} = \overline{\mu} \times \overline{\mathbf{B}}$$

$\overline{\mu}$ is the MAGNETIC DIPOLE MOMENT VECTOR

Magnitude of $\overline{\mu}$**:** $\mu = Iab$ = (Current)(Area)

or $\mu = NIab$ = (Number of turns)(Current)(Area)

if the loop has N turns of wire.

Direction of $\overline{\mu}$**:** This is given by the right hand rule:

Let the fingers curl in the direction of current flow.
The thumb then points along $\overline{\mu}$.

Ampere's Law

Recall: Gauss's Law relates $\overline{\mathbf{E}}$ to the charge distribution that produces it.

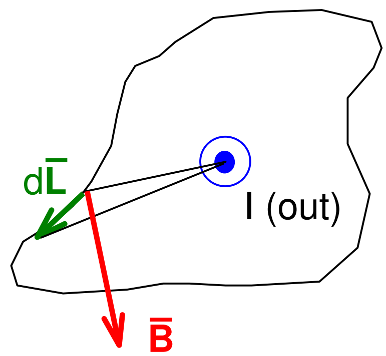
Likewise, Ampere's Law relates $\overline{\mathbf{B}}$ to the current distribution that produces it.

Recall: The magnetic field at perpendicular distance R from a long straight wire is

$$B = \frac{\mu_o I}{2\pi R}$$

View along the axis (with the current coming out of the page).

Consider an arbitrary closed path around the wire:

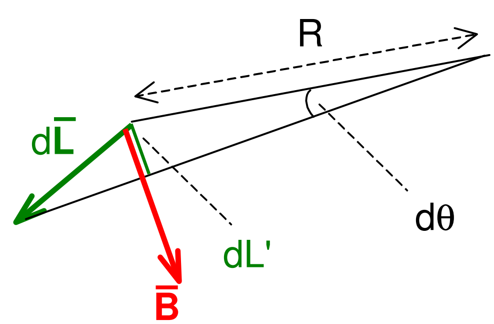


What is the line integral $\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}}$

around this closed path?

For a typical small element $d\overline{\mathbf{L}}$,

$$\overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = BdL' = BRd\theta = \frac{\mu_o I}{2\pi R}[Rd\theta]$$



$$\text{So} \quad \oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \frac{\mu_o I}{2\pi}\oint d\theta = \frac{\mu_o I}{2\pi}\int_0^{2\pi} d\theta = \frac{\mu_o I}{2\pi}2\pi$$

$$\Rightarrow \quad \oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I$$

This is AMPERE's LAW.

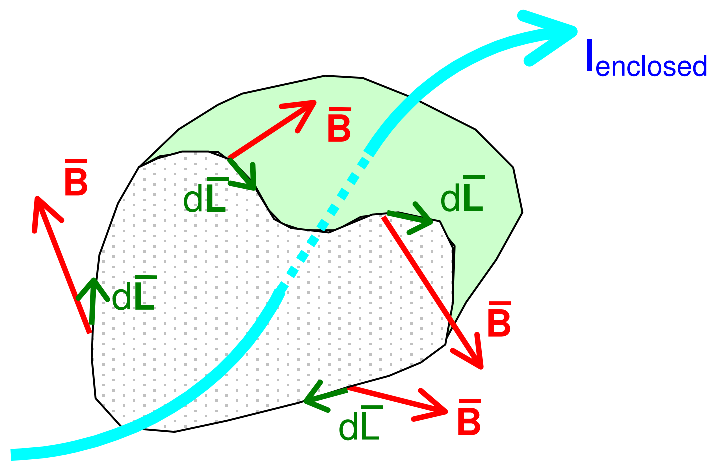
Note:

1. We carry out the integral along a some path around the conductor, NOT along the conductor and not necessarily along the field line.

2. Ampere's law holds for ANY closed path and ANY distribution of current.

3. The currrent, I, is the total current enclosed by the path, i.e., flowing through the area bounded by the path

or I is the current flowing through any "bag-like" surface whose edge is defined by the path.



Usually, we will write $\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I_{enclosed}$

4. Ampere's law $\Rightarrow$ $\overline{\mathbf{B}}$ is NOT conservative.

$$\oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = 0 \quad \text{but} \quad \oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} \neq 0$$

5. Ampere's Law in words: The line integral of the magnetic field around a closed path is equal to μ_o multiplied by the current enclosed by the path.

Procedure for using Ampere's Law

When to use it: When you want to find the magnetic field produced by some given distribution of current.

1. Draw a diagram showing the current distribution and the magnetic field that it produces.

   View along the axis of the current so that current is coming out of the page. This means that $\overline{\mathbf{B}}$ will be in the plane of the page.

2. Decide on the best CLOSED path for the line integral. Choose one that makes the integral easy

   ⇒ make B either parallel or perpendicular to dL

   In problems that we shall do, the path will always be either circular or rectangular.

3. Work out the line integral $\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}}$

4. Decide how much current flows THROUGH the area defined by the path.

5. Put $\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I_{enclosed}$ and rearrange the equation to find B as a function of position.

Examples:
1. $\overline{\mathbf{B}}$ due to a long straight wire.
2. $\overline{\mathbf{B}}$ due to a conducting cylinder
3. $\overline{\mathbf{B}}$ due to a long solenoid

See lecture notes.