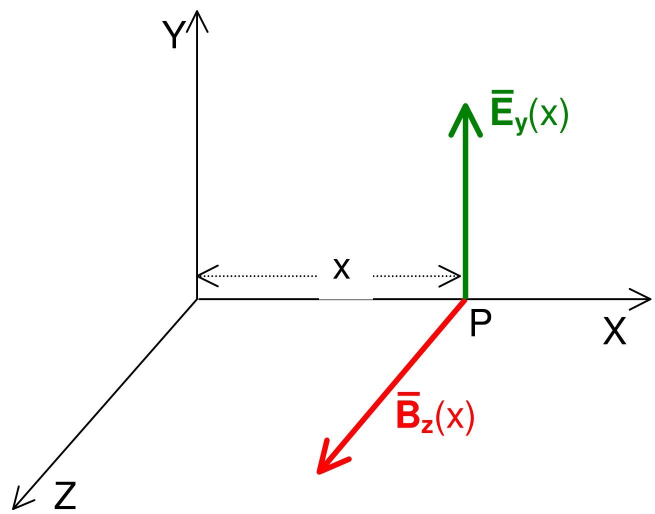
ELECTROMAGNETIC WAVES

Assume: A static pattern of $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ exists with $\overline{\mathbf{E}}$ in the y direction and $\overline{\mathbf{B}}$ in the z direction

- $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ both uniform
- $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ both in the y-z plane

E.g.: this could be due to a sheet of current parallel to the y-z plane, flowing in the y direction.

At some point, P, on the x axis, let $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ have the values indicated.



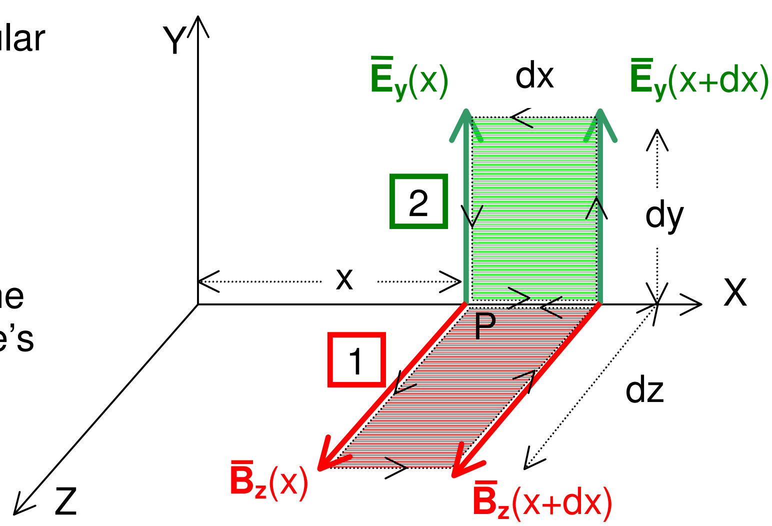
Let the situation change (e.g., the current changes).
What happens to $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ at point P?

Apply Maxwell's equations:

Faraday's Law for $\overline{\mathbf{E}}$
Ampere's Law for $\overline{\mathbf{B}}$

Consider two rectangular Loops, one in the x-y plane and one in the x-z plane

To begin with, apply the UNMODIFIED Ampere's Law to Loop 1:



$$\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I_{enc} = 0 \qquad \text{as current through the path} = 0$$

$$\Rightarrow \qquad [B_z(x)]dz - [B_z(x+dx)]dz = 0$$

So $$B_Z(x)dz - \left[B_Z(x) + \frac{\partial B_Z}{\partial x}dx\right]dz = 0 \quad \Rightarrow \quad -\frac{\partial B_Z}{\partial x}dxdz = 0$$

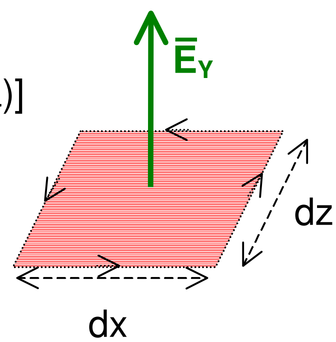
$$\Rightarrow \frac{\partial B_Z}{\partial x} = 0$$

⇒ ACTION AT A DISTANCE : INVALID ACCORDING TO SPECIAL RELATIVITY

SOLUTION: **Apply the** MODIFIED **version of Ampere's Law:**

$$\Rightarrow \quad -\frac{\partial B_z}{\partial x}dxdz = \mu_o I + \mu_o \varepsilon_o \frac{\partial \Phi}{\partial t} = 0 + \mu_o \varepsilon_o \frac{\partial [E_y dxdz]}{\partial t}$$

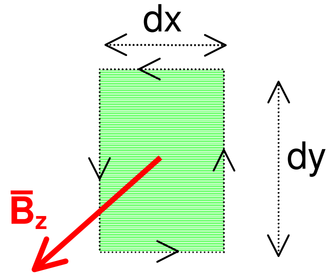
[because the electric flux through the loop = (E_y)(Area)]



$$\Rightarrow \boxed{\frac{\partial B_z}{\partial x} = -\mu_o \varepsilon_o \frac{\partial E_y}{\partial t}} \qquad \boxed{1}$$

i.e., **Time-varying $\bar{E}$ → position-varying $\bar{B}$**

Now apply FARADAY'S LAW to Loop 2:



$$\oint \bar{\mathbf{E}} \cdot d\bar{\mathbf{L}} = -\frac{d\Psi}{dt}$$

$$\Rightarrow \quad [E_y(x+dx)]dy - [E_y(x)]dy = -\frac{\partial [B_z dxdy]}{\partial t}$$

[because the magnetic flux through the loop = (B_z)(Area)]

$$\Rightarrow \quad \left[E_y(x) + \frac{\partial E_y}{\partial x}dx\right]dy - E_y(x)dy = -\frac{\partial [B_z dxdy]}{\partial t}$$

$$\Rightarrow \quad \boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \qquad \boxed{2}$$

i.e., **Time-varying $\overline{B}$ → position-varying $\overline{E}$**

Differentiate Equation1 with respect to t and Equation 2 with respect to x:

$$\frac{\partial}{\partial t}\frac{\partial B_z}{\partial x} = -\mu_o\varepsilon_o\frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial}{\partial x}\frac{\partial B_z}{\partial t} = -\frac{\partial^2 E_y}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial^2 E_y}{\partial t^2} = \frac{1}{\mu_o\varepsilon_o}\frac{\partial^2 E_y}{\partial x^2}$$

THE WAVE EQUATION
(see *Y&F* p. 601)

This describes a TRANSVERSE WAVE ($\overline{\mathbf{E}}$ perpendicular to direction of travel, X)

Now differentiate Equation 1 with respect to x and Equation 2 with respect to t:

$$\Rightarrow \quad \frac{\partial^2 B_z}{\partial t^2} = \frac{1}{\mu_o\varepsilon_o}\frac{\partial^2 B_z}{\partial x^2}$$

THE WAVE EQUATION AGAIN

Speed of propagation

$$\boxed{1} \qquad \frac{\partial B_z}{\partial x} = -\mu_o\varepsilon_o\frac{\partial E_y}{\partial t} \quad \Rightarrow \quad \frac{\partial x}{\partial t} = -\frac{1}{\mu_o\varepsilon_o}\frac{\partial B_z}{\partial E_y}$$

$$\boxed{2} \qquad \left[\frac{\partial x}{\partial t}\right]^2 = -\frac{1}{\mu_o\varepsilon_o} \quad \Rightarrow \quad \frac{\partial x}{\partial t} = \frac{1}{\sqrt{\mu_o\varepsilon_o}}$$

So, the SPEED OF ELECTROMAGNETIC WAVES is

$$c = \frac{1}{\sqrt{\mu_o\varepsilon_o}}$$

$\mu_o = 4\pi \times 10^{-7}$ H m^{-1}
$\varepsilon_o = 8.85 \times 10^{-12}$ F m^{-1}

$\Rightarrow c = 3.0 \times 10^{8}$ m s^{-1}

Solutions to the wave equation

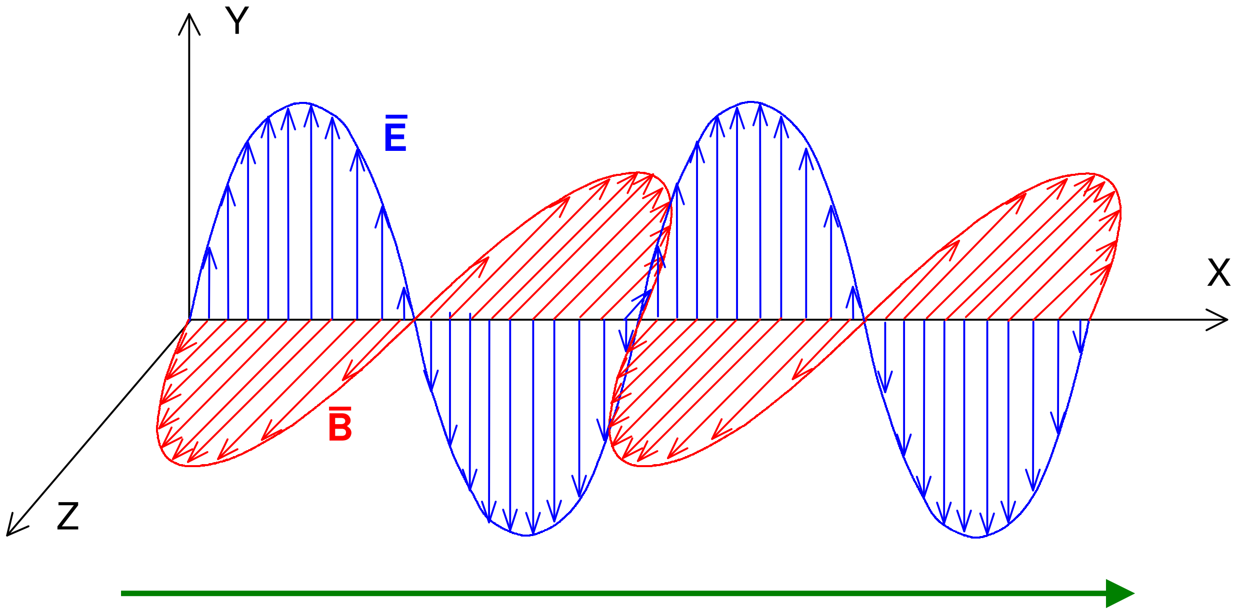
$$\frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2}$$

Solution is $E_y = E_o \cos(\omega t - \beta x)$

$$\left.\begin{array}{l} \dfrac{\partial^2 E_y}{\partial t^2} = \omega^2 E_o \cos(\omega t - \beta x) \\[2ex] \dfrac{\partial^2 E_y}{\partial x^2} = \beta^2 E_o \cos(\omega t - \beta x) \end{array}\right\} \quad \omega^2 = c^2\beta^2 \Rightarrow \beta = \omega/c$$

So $E_y = E_o \cos\left[\omega\left(t - \frac{x}{c}\right)\right]$ and $B_z = B_o \cos\left[\omega\left(t - \frac{x}{c}\right)\right]$

Period of oscillation: $T = \frac{2\pi}{\omega}$



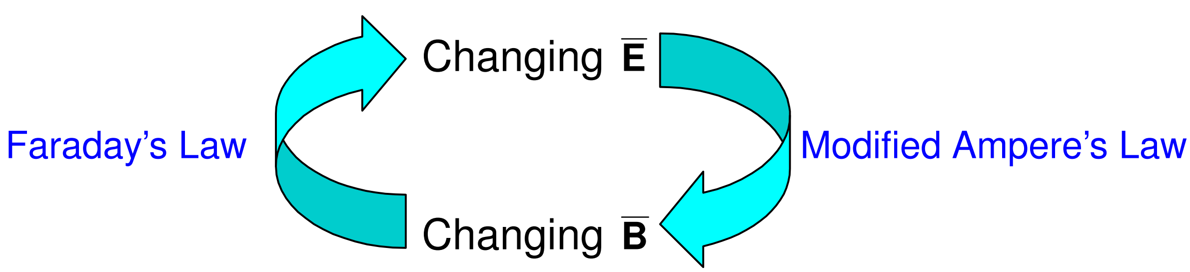
The whole pattern moves in x-direction with speed c

Note:

1. $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ are perpendicular to each other

2. $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ are perpendicular to the direction of travel

3. The wave is self-sustaining:



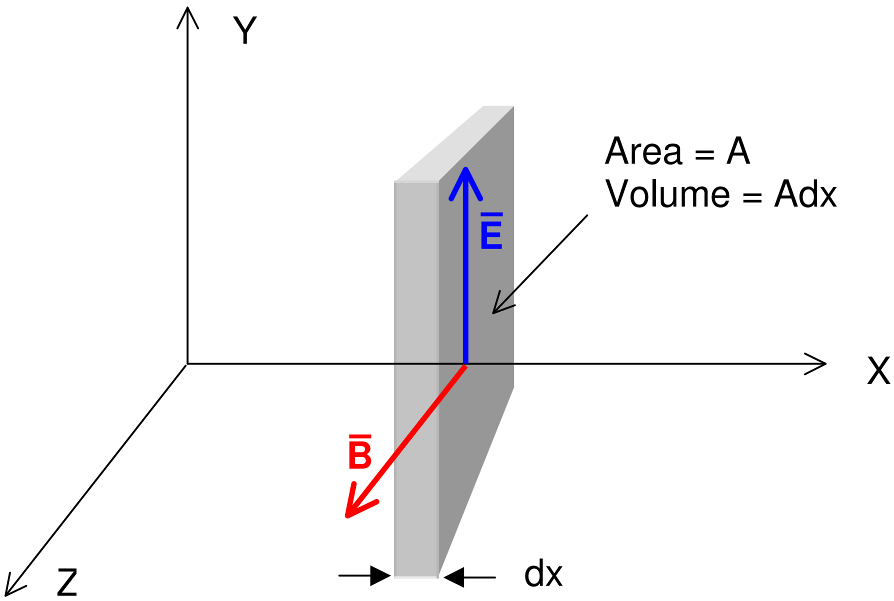
4. $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ contain energy $\Rightarrow$ the wave transports energy.

Power flow in a plane electromagnetic wave

Recall: The energy densities (energy per unit volume) of the electric and magnetic fields are

$$u_E = \frac{1}{2}\varepsilon_o E^2 \qquad u_B = \frac{1}{2}\frac{B^2}{\mu_o}$$

Consider a plane wave propagating in the x-direction, and evaluate the energy contained in a thin slab of area A, thickness dx:



The electromagnetic energy in the slab is: $dU = \frac{1}{2}\left[\varepsilon_o E^2 + \frac{B^2}{\mu_o}\right]Adx$

So $dU = \frac{1}{2}\left[\varepsilon_o E^2 + \frac{E^2}{c^2\mu_o}\right]Adx$ But $c = \frac{1}{\sqrt{\mu_o\varepsilon_o}} \Rightarrow \varepsilon_o = \frac{1}{\mu_o c^2}$

Therefore $dU = \frac{E^2}{c^2\mu_o}Adx$ or $dU = \frac{1}{\mu_o c}(EB)Adx$

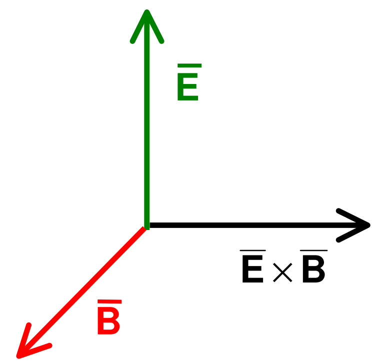
But $\frac{\partial x}{\partial t} = -\frac{\partial E}{\partial B} = \frac{E}{B} = c \qquad \Rightarrow \qquad E = cB$

Power crossing area A is P = dU/dt:

$$P = \frac{1}{\mu_o} EBA \qquad \text{as} \qquad c = \frac{dx}{dt} \qquad \Rightarrow \qquad \text{Power per unit area} = \frac{EB}{\mu_o}$$

Direction of power flow = direction of $\overline{\mathbf{E}} \times \overline{\mathbf{B}}$

Magnitude of power flow $= \dfrac{EB}{\mu_o}$



$\Rightarrow$ Power flow per unit area is given by

$$\overline{\mathbf{S}} = \frac{1}{\mu_o}\left(\overline{\mathbf{E}} \times \overline{\mathbf{B}}\right)$$

is the POYNTING VECTOR

Which carries more energy, $\overline{E}$ or $\overline{B}$?

$$u_E = \frac{1}{2}\varepsilon_o E^2 \qquad u_B = \frac{1}{2}\frac{B^2}{\mu_o} \qquad \text{But } E = cB$$

$$\text{So} \quad E^2 = \frac{B^2}{\mu_o \varepsilon_o} \qquad \Rightarrow \qquad u_E = \frac{1}{2}\varepsilon_o \frac{B^2}{\mu_o \varepsilon_o} = \frac{1}{2}\frac{B^2}{\mu_o} = u_B$$

So the two contributions are equal – the wave is a continuing exchange of energy between the electric and magnetic fields.

The fact that, for an electromagnetic wave, E = cB, does not imply that E is more “important” than B in electromagnetic radiation. The value of c is only large as a consequence of the definition of the metre in the SI system. The most natural system of units is one in which c = 1, but that would not be very suitable for everyday applications.