MAXWELL'S EQUATIONS

Maxwell's four equations

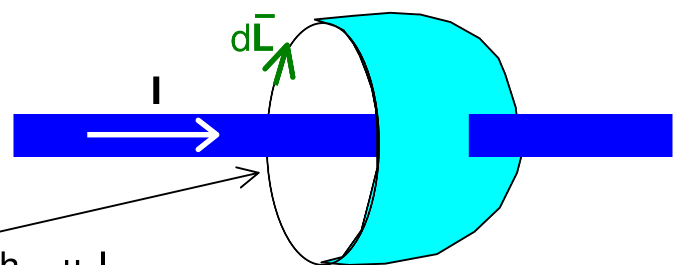
In the 1870's, James Clerk Maxwell showed that four equations constitute a complete description of the electric and magnetic fields, or **THE ELECTROMAGNETIC FIELD**

- Gauss's Law for $\overline{\mathbf{E}}$
- Gauss's Law for $\overline{\mathbf{B}}$
- Faraday's Law
- Maxwell's modified version of Ampere's Law

(The first three: We've met these three already)

Why does Ampere's law need to be modified?

(a) Apply it to a continuous conductor:



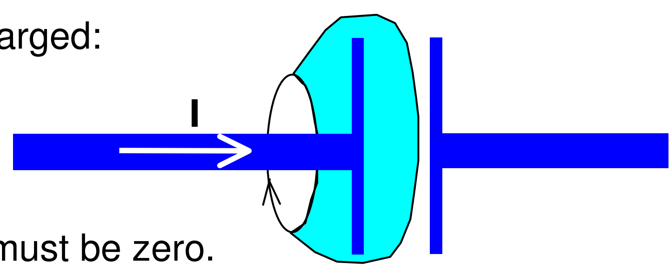
$\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}}$ around this path $= \mu_o I$

I = current passing through an arbitrary bag-like surface whose edge is the path.

$\overline{\mathbf{B}}$ = magnetic field created by the moving charges in the wire

(b) Apply it to a capacitor being charged:

Let the bag-like surface pass **BETWEEN** the plates



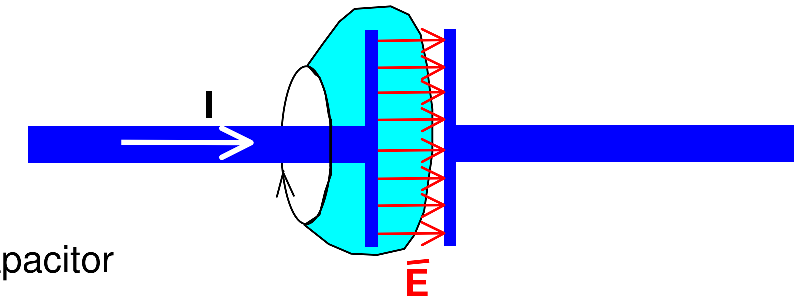
$\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I_{enc} = 0 \quad \Rightarrow \quad \overline{\mathbf{B}}$ must be zero.

So, Ampere's Law $\Rightarrow \overline{\mathbf{B}} = 0$.

But clearly $\overline{\mathbf{B}} \neq 0$ because charges are in motion.

Maxwell's modification

The surface does not intercept any current, but it DOES intercept ELECTRIC FLUX



How much flux is intercepted ?

Let Q = charge on capacitor

$$\Phi = \frac{Q}{\varepsilon_o} \Rightarrow \frac{d\Phi}{dt} = \frac{I}{\varepsilon_o} \Rightarrow I = \varepsilon_o \frac{d\Phi}{dt}$$

$\Rightarrow$ If we want $\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I$ as for a straight wire, we must claim that

$$\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I + \mu_o \varepsilon_o \frac{d\Phi}{dt}$$

MAXWELL'S 4th EQUATION

Term 1 ($\mu_o I$), Term 2 ($\mu_o \varepsilon_o \frac{d\Phi}{dt}$)

Continuous wire : Term 1 = $\mu_o I$, Term 2 = 0 } OK

Capacitor : Term 1 = 0, Term 2 = $\mu_o \varepsilon_o \frac{d\Phi}{dt} = \mu_o I$ } OK

Maxwell showed that this is a general relation which holds ALWAYS.

Note: The modified Ampere's Law can be written as

$$\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o (I + I_d) \qquad I_d = \varepsilon_o \frac{d\Phi}{dt} = \text{DISPLACEMENT CURRENT}$$

What does this new version of Ampere's law imply about the relationship between $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$?

$$\Phi \equiv (E)(\text{Area}) \quad \Rightarrow \quad \text{if } \frac{d\Phi}{dt} \neq 0 \quad \text{then } \frac{dE}{dt} \neq 0$$

i.e., $\frac{d\Phi}{dt}$ represents a CHANGING ELECTRIC FIELD.

If $\frac{d\Phi}{dt} \neq 0$ then $\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} \neq 0$ and therefore $\overline{\mathbf{B}} \neq 0$.

i.e., if there is a changing electric field, then the magnetic field cannot be zero

or **A CHANGING ELECTRIC FIELD PRODUCES A MAGNETIC FIELD**

Recall: Faraday's Law: $\oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = -\frac{d\Psi}{dt}$

This causes this

Changing $\overline{\mathbf{B}}$ causes $\overline{\mathbf{E}}$

Now, the modified version of Ampere's Law implies that **the reverse is also true**:

$$\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I + \mu_o \varepsilon_o \frac{d\Phi}{dt}$$

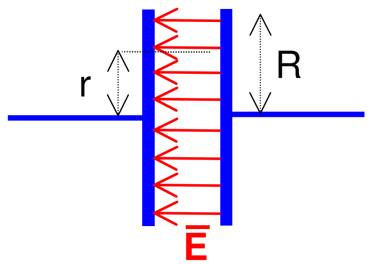
This causes this

Changing $\overline{\mathbf{E}}$ causes $\overline{\mathbf{B}}$

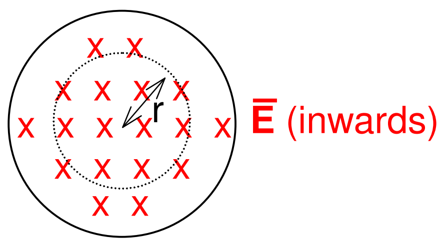
Example: A parallel plate capacitor, radius R, is connected to a source of alternating emf.

Alternating emf → alternating electric field $E = E_o \sin(\omega t)$

What is the magnetic field (i) Inside the capacitor (r < R) ?
(i) Outside the capacitor (r > R) ?



SIDE VIEW

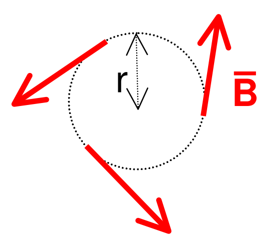


TOP VIEW

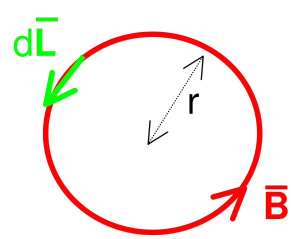
The current flowing across the capacitor, I = 0 (plates are separated by a vacuum or insulator).

(i) r < R: Symmetry ⇒ the magnetic field has the same magnitude and direction at all points on the dotted circle of radius r

Direction of $\bar{\mathbf{B}}$ will be tangential because the field is associated with current flowing ⊥ to the plane of the path.



Apply modified Ampere's Law to the circular path:



Integrate $\bar{\mathbf{B}} \cdot d\bar{\mathbf{L}}$ around the path.
$d\bar{\mathbf{L}}$ is parallel to $\bar{\mathbf{B}}$ everywhere along the path,
And B is also the same all around the path.

$$\text{So} \oint \bar{\mathbf{B}} \cdot d\bar{\mathbf{L}} = \int_0^{2\pi r} B\,dL = B\int_0^{2\pi r} dL = B(2\pi r)$$

The electric flux through the path is $\Phi = (\text{Field})(\text{Area}) = E_o \sin(\omega t)\pi r^2$

$$\mu_o I + \mu_o\varepsilon_o \frac{d\Phi}{dt} = 0 + \mu_o\varepsilon_o \frac{d(\pi r^2 E_o \sin(\omega t))}{dt}$$

So $$2\pi r B = \pi r^2 \mu_o \varepsilon_o \omega E_o \cos(\omega t)$$

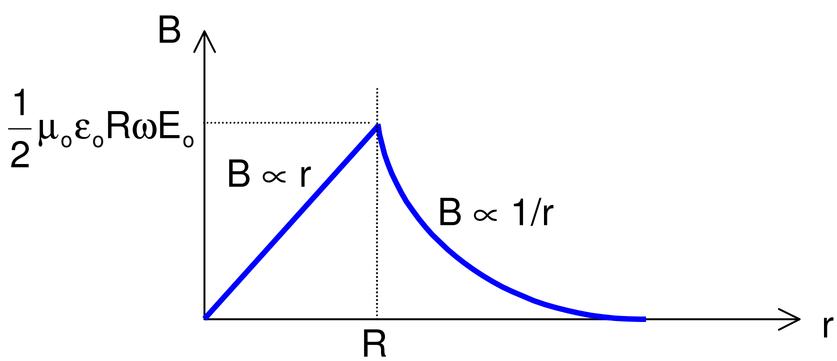
$$\Rightarrow \quad B(t) = \frac{1}{2}\mu_o\varepsilon_o r\omega E_o \cos(\omega t) \quad \text{or} \quad \boxed{B(t) = B_o \cos(\omega t)}$$

(i) r > R: The same analysis applies except that

$\Phi = \pi R^2 E_o \sin(\omega t)$ (no contribution from the area outside R)

$$\Rightarrow \quad \boxed{B(t) = \frac{1}{2}\frac{\mu_o\varepsilon_o R^2 \omega E_o}{r}\cos(\omega t)}$$

Sketch of the amplitude of the magnetic field vs. radius:



Summary of Maxwell's Equations (in integral form)

① $\oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = \frac{Q_{enclosed}}{\varepsilon_o}$ **Gauss's Law for the Electric Field**

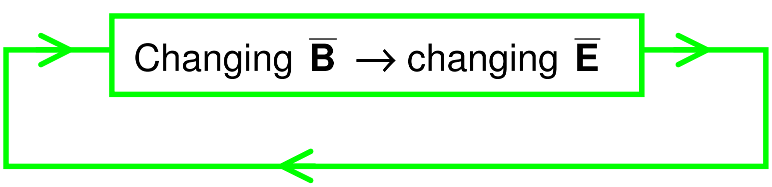
② $\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{A}} = 0$ **Gauss's Law for the Magnetic Field**

③ $\oint \overline{\mathbf{E}} \cdot d\overline{\mathbf{L}} = -\frac{d\Psi}{dt}$ **Faraday's Law**

④ $\oint \overline{\mathbf{B}} \cdot d\overline{\mathbf{L}} = \mu_o I + \mu_o \varepsilon_o \frac{d\Phi}{dt}$ **Maxwell's modification of Ampere's Law**

NB: ③ $\Rightarrow$ Changing $\overline{\mathbf{B}}$ generates $\overline{\mathbf{E}}$

④ $\Rightarrow$ Changing $\overline{\mathbf{E}}$ generates $\overline{\mathbf{B}}$



$\Rightarrow$ **OSCILLATION OF ENERGY BETWEEN THE ELECTRIC AND MAGNETIC FIELDS**

In fact, Maxwell's equations imply the existence of

ELECTROMAGNETIC WAVES