Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-304 Physical Dynamics

Time Allowed: 2 hours 15 minutes

Date: 19th May 2008

Time: 10:00

Instructions: Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 40 marks, each question in section B carries 30 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

NUMERIC CALCULATORS ARE PERMITTED IN THIS EXAMINATION.

Data: A formula sheet containing mathematical results that may be of help in various questions is provided at the end of the examination paper.

Examiners: Dr G Travaglini (CO)
Prof WJ Spence (DCO)

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

SECTION A: Answer ALL questions in this section

A1.

Consider the motion of a single particle under the influence of a force $\mathbf{F}$.

(i) Write down the expression for the work $W[C_{12}]$ done by the force $\mathbf{F}$ when the particle moves from the initial position $\mathbf{r}_1$ to the final position $\mathbf{r}_2$ along a certain path C_{12}. [3]

(ii) Define what a conservative force is, and write down the expression for the energy of the particle assuming that $\mathbf{F}$ is conservative. Show that the time derivative of the energy calculated for a solution of the equations of motion is zero. [5]

(iii) Again assuming that the force $\mathbf{F}$ is conservative, show that the work done by $\mathbf{F}$ as the particle moves from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ only depends on the initial and final positions but not on the path C_{12} joining them. [4]

A2.

For each of the following three-dimensional mechanical systems, describe in words what are the conserved quantities, and explain what are the corresponding Noether symmetries responsible for their conservation.

(i) Free particle. [3]

(ii) Particle moving in a potential $V(x, z)$, i.e. a potential independent of the coordinate y of the particle and of time, but otherwise generic. [4]

(iii) Particle moving in a potential with spherical symmetry, $V = V(|\mathbf{r}|)$, where $\mathbf{r} := (x, y, z)$ denotes, as usual, the position vector of the particle. [4]

(iv) Particle moving in a gravitational field oriented along the z axis, and constrained to move on the surface of a cylinder with symmetry axis coincident with the z axis. [4]

 Question continues over page

A3.

Consider the motion of a particle with one degree of freedom parameterised by a coordinate q, in a potential $V(q)$.

(i) Define the Lagrangian of the particle $L(q, \dot{q})$, and give the momentum conjugate to q. [4]

(ii) Explain how the Hamiltonian $H(p, q)$ is obtained from the Lagrangian. [4]

(iii) Derive the Hamilton equations for the particle. [5]

SECTION B: Answer ONLY TWO QUESTIONS from this section

B1.

Consider the motion of a single particle of mass m subject to a central potential $V(r)$, where $\mathbf{r}$ is the position vector of the particle in an inertial system $(\hat{x}, \hat{y}, \hat{z})$ with origin at O, and $r := |\mathbf{r}|$.

(i) Prove that the angular momentum $\mathbf{L}$ of the particle about the point O is a conserved quantity. [7]

(ii) The direction of the constant vector $\mathbf{L}$ determines the plane of the orbit. Choosing plane polar coordinates (r, ϕ) on this plane as generalised coordinates, write down the Lagrangian of the system. [5]

(iii) Using the Lagrange equations, prove that the momentum p_ϕ associated to the coordinate ϕ is conserved. What physical quantity does it correspond to? [5]

(iv) Write down the expression for the energy of the particle, and use the conservation of p_ϕ to eliminate $\dot{\phi}$ from this expression, thus giving the energy in the form

$$E(r, \dot{r}) = \frac{1}{2} m \dot{r}^2 + V_{\text{eff}}(r) .$$

Calculate the one-dimensional effective potential $V_{\text{eff}}(r)$. [7]

(v) For the case of the gravitational potential $V(r) = -k/r$, $k > 0$, sketch the graph of the effective potential $V_{\text{eff}}(r)$, and determine the position r_{m} of its minimum. What kind of orbit does this minimum correspond to? [6]

B2.

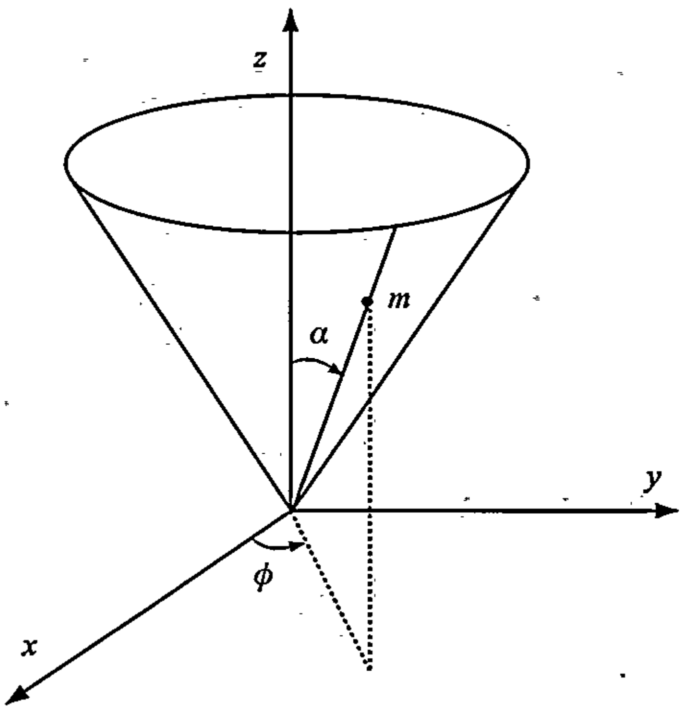
A particle of mass m is constrained to move on the surface of the inverted circular cone of opening semi-angle α represented in the figure. Gravity acts as usual along the vertical direction.

(i) How many degrees of freedom does the system have? [4]

(ii) Picking spherical coordinates, write down the Lagrangian of the system and the Lagrange equations. [10]

(iii) Show that the momentum p_ϕ associated to the azimuthal angle ϕ is equal to the z-component L_z of the angular momentum. What is the symmetry responsible for its conservation? [8]

(iv) By eliminating the coordinate ϕ and its time derivative, obtain the energy as a function of r and $\dot{r}$ only, in the form $E(r,\dot{r}) = (1/2)m\dot{r}^2 + V_{\text{eff}}(r)$. Obtain this effective one-dimensional potential $V_{\text{eff}}(r)$, and calculate the value r_{m} of r which minimises it. Explain what physical situation the minimum of the potential corresponds to. [8]



B3.

Consider the motion of a particle of mass m described by the following Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + k(x\dot{y} - y\dot{x}) .$$

(i) Write down the Lagrange equations, and use them to show that the three quantities

$$\mathcal{I}_x := m\dot{x} - 2ky , \qquad \mathcal{I}_y := m\dot{y} + 2kx , \qquad \mathcal{I}_z := m\dot{z} ,$$

are conserved. [8]

(ii) Show that the Lagrangian L is exactly invariant (i.e. $\delta L = 0$) under a translation along the $\hat{z}$ axis, and invariant under translations along the $\hat{x}$ and $\hat{y}$ axis up to a total time derivative of a function. [9]

(iii) Show that L is invariant under a rotation by an infinitesimal angle ϵ about the $\hat{z}$ axis,

$$\delta x = \epsilon y , \qquad \delta y = -\epsilon x .$$

Show that the corresponding Noether invariant is given by

$$\tilde{\mathcal{I}} := m(x\dot{y} - y\dot{x}) + k(x^2 + y^2) .$$

[9]

(iv) Use the results derived in point (i) above to show that the quantity $v_\perp := \dot{x} + i\dot{y}$ satisfies the equation

$$\frac{d}{dt}v_\perp = -i\omega v_\perp ,$$

where $\omega := 2k/m$. [4]

B4.

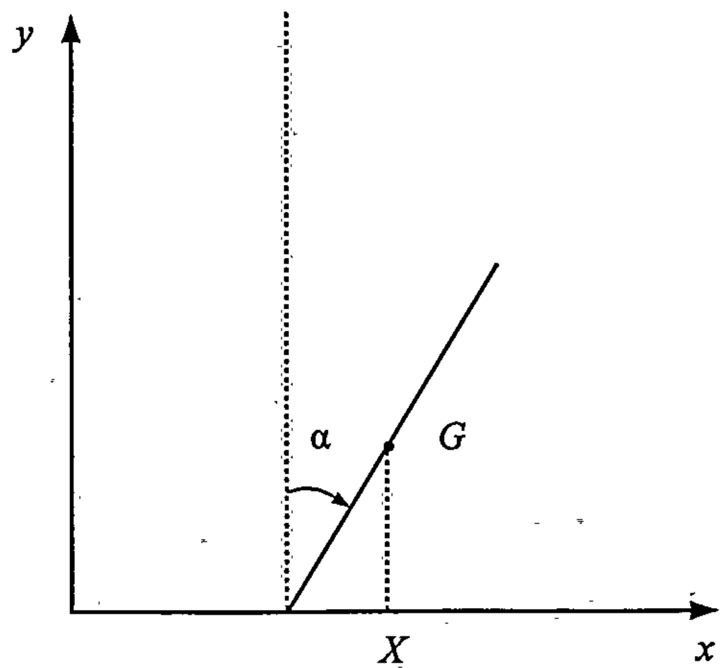
A rigid rod of length $2L$ has its lower end in contact with a horizontal plane. The mass of the rod is equal to m, and its mass density is uniform. Gravity acts as usual in the vertical direction. The rod is initially at an angle α with respect to the vertical (see the figure), when it is released from rest. The motion is constrained to take place in the vertical plane $(\hat{x}, \hat{y})$.

(i) How many degrees of freedom does the system have? [6]

(ii) Calculate the moment of inertia of the rod with respect to an axis orthogonal to the rod and passing through its centre of mass G. [8]

(iii) Choosing appropriate generalised coordinates, write down the Lagrangian of the system. [10]

(iv) Show that in the subsequent motion after the rod is released from rest, the x-coordinate X of the centre of mass G remains constant. [6]



FORMULA SHEET

Plane polar coordinates:

$$
\begin{aligned}
\mathbf{r} &:= r\hat{\mathbf{r}}\,, \\
\dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}\,, \\
\dot{\mathbf{r}}^2 &= \dot{r}^2 + r^2\dot{\phi}^2\,.
\end{aligned}
$$

Spherical coordinates:

$$
\begin{aligned}
\mathbf{r} &:= r\hat{\mathbf{r}}\,, \\
\dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\hat{\boldsymbol{\phi}}\,, \\
\dot{\mathbf{r}}^2 &= \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\,.
\end{aligned}
$$

End of Examination Paper
Dr G Travaglini