Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-304 Physical Dynamics

Time Allowed: 2 hours 15 minutes

Date: 8th May 2007

Time: 10:00

Instructions: Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 40 marks, each question in section B carries 30 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

NUMERICAL CALCULATORS ARE PERMITTED IN THIS EXAMINATION.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

SECTION A: Answer ALL questions in this section

A1.

Consider a system of n particles of masses m_i, $i = 1, \ldots, n$ which are subject to external and internal forces; let $\mathbf{F}_i^{(e)}$ be the external force acting on particle i, and $\mathbf{F}_{ij}$ the internal force due to particle i on particle j.

(i) Write the Newton equations for the system of particles. [2]

(ii) Show that the total momentum $\mathbf{P} := \sum_{i=1}^{n} \mathbf{p}_i$ is conserved if the sum of all external forces is zero, $\sum_{i=1}^{n} \mathbf{F}_i^{(e)} = 0$, and the weak form of the action and reaction principle holds (the weak form of the action and reaction principle states that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$). [4]

(iii) Define what is the coordinate $\mathbf{R}$ of the centre of mass of the system of particles. [2]

(iv) Show that the total angular momentum of the system about the origin $\mathbf{L} := \sum_{i=1}^{n} \mathbf{L}_i$ can be rewritten as $\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_{i=1}^{n} \mathbf{L'}_i$, where $\mathbf{P}$ is the total momentum and $\mathbf{L'}_i := \mathbf{r'}_i \times \mathbf{p'}_i$ is the relative angular momentum of particle i, with $\mathbf{r'}_i := \mathbf{r}_i - \mathbf{R}$ and $\mathbf{p'}_i := m_i \dot{\mathbf{r}}'_i$. [5]

A2.

Consider a single particle of mass m moving in one dimension parametrised by the coordinate x. The particle is subject to a conservative force whose potential is $V(x)$.

(i) Write down the Lagrangian of the system, L, and state Hamilton's principle of least action. [2]

(ii) Write down the Lagrange equations for the system, and use them to show that, if $V(x) = 0$, then the momentum of the particle is a conserved quantity. In the case $V(x) = 0$, say what is the symmetry that guarantees that momentum is conserved, as a consequence of Noether's theorem. [5]

(iii) Consider the energy $E(x, \dot{x}) := (1/2)m\dot{x}^2 + V(x)$. Calculate explicitly the time derivative $\dot{E}(x, \dot{x})$ of the energy, showing that it vanishes upon using the Lagrange equations. [4]

(iv) Consider two Lagrangians, L and $\widetilde{L}$, differing by the time derivative of a generic function $F := F(q, t)$ of the generalised coordinate q and of time; that is, $\widetilde{L}(q, \dot{q}) := L(q, \dot{q}) + dF/dt$. Using Hamilton's principle of least action, explain why L and $\widetilde{L}$ give rise to the same equations of motion. [3]

Question continues over page

A3.

Consider the motion of two particles of masses m_1 and m_2 in three-dimensional space, subject to a central potential $V = V(r)$, where $r := |\mathbf{r}_1 - \mathbf{r}_2|$, and $\mathbf{r}_1$, $\mathbf{r}_2$ are the position vectors of the two particles in an inertial reference frame, respectively.

(i) Write the Lagrangian of the system $L(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)$. [2]

(ii) Write the Lagrange equations. [3]

(iii) Explain why we say that the $V(r)$ is a *central* potential. [3]

(iv) Introduce the centre of mass coordinate $\mathbf{R}$ and the relative coordinate $\mathbf{r} := \mathbf{r}_2 - \mathbf{r}_1$. Show that the Lagrangian of the system in terms of the generalised coordinates $\mathbf{R}$, $\mathbf{r}$ and their time derivatives $\dot{\mathbf{R}}$ and $\dot{\mathbf{r}}$ is equal to $L_{\text{total}} = L_{\text{free}}(\mathbf{R}, \dot{\mathbf{R}}) + L(\mathbf{r}, \dot{\mathbf{r}})$, where $L_{\text{free}} = (1/2)(m_1 + m_2)\dot{\mathbf{R}}^2$, and $L(\mathbf{r}, \dot{\mathbf{r}}) = (1/2)\mu\dot{\mathbf{r}}^2 - V(r)$, where $\mu := m_1 m_2/(m_2 + m_2)$ is the reduced mass. [5]

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SECTION B: Answer ONLY TWO QUESTIONS from this section

B1.

(i) State Noether's theorem for a Lagrangian system described by a set of generalised coordinates $\mathbf{q}$. [4]

(ii) Prove that the expression for the conserved quantity $\delta\mathcal{I}$ for a system described by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ that is exactly invariant ($\delta L = 0$) under a transformation $\delta\mathbf{q}$ of the generalised coordinates is given by

$$\delta\mathcal{I} = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \delta\mathbf{q}$$

(as usual, a dot between two vectors stands for the scalar product). [10]

(iii) Consider the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(y, z) .$$

What are the conserved quantities, and what the symmetries responsible for their conservation? [7]

(iv) Consider the Lagrangian of a particle moving in three dimensions written in terms of cylindrical coordinates (ρ, ϕ, z),

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\phi, z) .$$

The potential V does not depend on ρ. Furthermore, the dependence on ϕ and z in V occurs only through the combination $h\phi + z$, where h is a constant number with dimensions of a length. In other words, we assume that

$$V(\phi, z) := f(h\phi + z) ,$$

where f is an arbitrary function.
Check that any transformation for which $\delta z = -h\,\delta\phi$, $\delta\rho = 0$, is a symmetry of the system, and show that the associated conserved quantity $\mathcal{I}$ is given by

$$\mathcal{I} = hp_z - L_z ,$$

where p_z and L_z are the momenta associated to z and ϕ, respectively. [9]

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B2.

A simple pendulum of mass m and length l is constrained to move in a vertical plane. The point of suspension has a mass M and can move along a horizontal axis. Gravity acts as usual along the vertical direction.

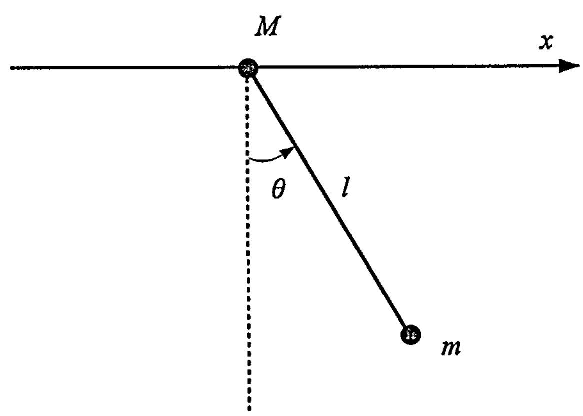
(i) How many degrees of freedom does the system have? [3]

(ii) Write the Lagrangian of the system in terms of an appropriate set of generalised coordinates. [10]

(iii) Let x be the coordinate parametrising the position of the suspension point along the horizontal axis. Show that the coordinate x is cyclic, and calculate the momentum associated to x. [4]

(iv) What is the symmetry responsible for the conservation of the momentum associated to x? [3]

(v) Find the frequency of small oscillations of the pendulum around the equilibrium position $\theta = 0$ (the angle θ is defined in the figure). [10]



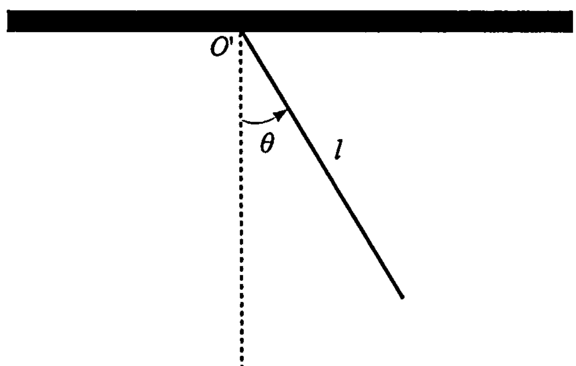
B3.

(i) Consider a rigid body which is free to oscillate about a fixed point O'. Show that the angular momentum $\mathbf{L}_{O'}$ about the fixed point and the angular velocity vector ω are related by the equation $\mathbf{L}_{O'} = \mathbf{I}_{O'}\omega$, where $\mathbf{I}_{O'}$ is the inertia tensor. [8]

(ii) Calculate the inertia tensor of a linear rigid rod of mass m and length l with respect to an origin at one of the two ends of the rod (you could orient the rod along the z-axis, with the origin at one of the two ends of the rod). Assume that the width of the rod is negligible compared to the length, so that you can effectively treat it as one-dimensional. The density of the rod is constant. [8]

(iii) The rod is free to oscillate in a vertical plane about a fixed axis orthogonal to this plane. The axis intersects the plane at O', which coincides with the position of one of the two extrema of the rod (see figure). Gravity acts along the vertical direction. Find the Lagrangian of the system. [10]

(iv) Determine the frequency of small oscillation of the system about the equilibrium position $\theta = 0$ (the angle θ is defined in the figure). [4]



B4.

(i) Define the Hamiltonian $H(\mathrm{p}, \mathrm{q})$ of a mechanical system and explain how it is derived from the Lagrangian of the system $L(\mathrm{q}, \dot{\mathrm{q}})$. [5]

(ii) Derive the Hamilton equations for $H(\mathrm{p}, \mathrm{q})$. [5]

(iii) Let A be a generic function of p, q, $A = A(\mathrm{p}, \mathrm{q})$ (we assume that A does not depend explicitly on time, but only implicitly through p and q). Using the Hamilton equations, show that the time derivative of A is

$$\dot{A} = \frac{\partial A}{\partial \mathrm{q}} \cdot \frac{\partial H}{\partial \mathrm{p}} - \frac{\partial A}{\partial \mathrm{p}} \cdot \frac{\partial H}{\partial \mathrm{q}}$$

(as usual, a dot between two vectors stands for the scalar product). [5]

(iv) Consider the motion of a particle of mass m on a plane, subject to the central potential $V = V(r)$. Write the Lagrangian of the system in plane polar coordinates (r, ϕ), and derive the corresponding Hamiltonian. [5]

(v) Using the result of (iii) above, show that $\dot{p}_\phi = 0$ (p_ϕ is the momentum associated to ϕ). [5]

(vi) Show that $p_\phi = L_z$, i.e. p_ϕ is the angular momentum of the particle about the origin. What is the symmetry responsible for its conservation? [5]

End of Examination Paper **Dr G Travaglini**

