Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-304 Physical Dynamics

Time Allowed: 2 hours 15 minutes

Date: May 5, 2006

Time: 10:00

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 40 marks, each question in section B carries 30 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.**

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

NUMERIC CALCULATORS ARE PERMITTED IN THIS EXAMINATION.

Data: A formula sheet which may help in answering some of the questions is included at the end of the paper.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

 version PhD

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SECTION A: Answer **ALL** questions in Section A.

A1 A system of N point masses m_i, located at positions $\boldsymbol{r}_i$, $i = 1, \ldots, N$, with respect to an inertial frame, moves under the influence of external and internal forces where $\boldsymbol{F}_i^{(e)}$ is the external force on particle i and $\boldsymbol{F}_{ij}$ is the two-body force on particle i due to particle j.

Using Newton's Second Law applied to each particle in an inertial frame of reference, and making appropriate use of Newton's Third law, derive the result

$$\frac{d\boldsymbol{P}_{tot}}{dt} = \boldsymbol{F}_{tot}^{(e)} \quad ,$$

where $\boldsymbol{P}_{tot}$ is the total linear momentum of the system and $\boldsymbol{F}_{tot}^{(e)}$ is the net external force on the system. [7]

A2 Define the position $\boldsymbol{R}$ of the centre of mass of this system. [2]

A3 If M denotes the total mass of the system show that [2]

$$\boldsymbol{P}_{tot} = M\frac{d\boldsymbol{R}}{dt} \quad .$$

A4 What conditions on the forces acting on the particles are sufficient for the total angular momentum of such a system to be conserved? [3]

A5 A conservative mechanical system is described by a set of generalised coordinates q_i, $i = 1, \ldots, N$. Define the Lagrangian for such a system and state Lagrange's equations in terms of the generalised coordinates. [3]

A6 Define what is meant by the generalised momentum p_i and the generalised force Q_i conjugate to the generalised coordinate q_i. [2]

A7 Explain what condition on the Lagrangian is sufficient for the generalised momentum p_i to be conserved. [1]

A8 A spherical pendulum is constructed of a point mass M suspended under gravity from a fixed frictionless pivot by a light inextensible rod of length L. How many degrees of freedom does the pendulum have? [2]

A9 What coordinate system matches the degrees of freedom in question A8? [2]

A10 Using this set of coordinates, write down the Lagrangian function L for the pendulum and obtain Lagrange's equations of motion. [8]

A11 Give expressions for all the conserved quantities for the pendulum and state what symmetries correspond to each of them. [4]

A12 If the pendulum has a non-zero value of its angular momentum about the vertical axis through the pivot, show that the mass M will never pass through this vertical axis in its subsequent motion. [4]

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SECTION B:

Answer **TWO** questions only from this Section.

B1

(a) Define what is meant by the *action* for a mechanical system. State Hamilton's principle for a mechanical system with one degree of freedom and use it to derive the Lagrange equation of motion for the system. [10]

(b) Explain how the energy is defined in the Lagrangian formalism and show that it is conserved if the Lagrangian L has no explicit time dependence. Without detailed derivations, explain how the Hamiltonian function H is obtained for a conservative mechanical system. Treating the sun as fixed, and assuming that a comet of mass M_c moves in a plane subject only to the sun's gravitational field, obtain the Hamiltonian function for the comet in plane polar coordinates r, θ. [12]

(c) Define what is meant by *phase space* and by a *phase trajectory*. The comet above moves in an orbit whose total energy is zero. Discuss the phase trajectory of the comet in the $p_r - r$ phase plane. [8]

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B2

(a) In an inertial frame of reference a rigid body rotates at angular velocity $\boldsymbol{\omega}$ about a fixed axis.

i) Taking an origin on the axis of rotation show that the kinetic energy of the rigid body can be expressed as $T = \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{\omega}$. [4]

ii) Show also that the angular momentum of the body $\boldsymbol{L}$ is related to its angular velocity $\boldsymbol{\omega}$ by a linear equation of the form $\boldsymbol{L} = \boldsymbol{I}\cdot\boldsymbol{\omega}$ where the moment of inertia tensor $\boldsymbol{I}$ is represented as a three by three matrix. Give explicit expressions for the elements of this matrix. [8]

iii) Explain what is meant by a *principal axis* system for a rigid body. [2]

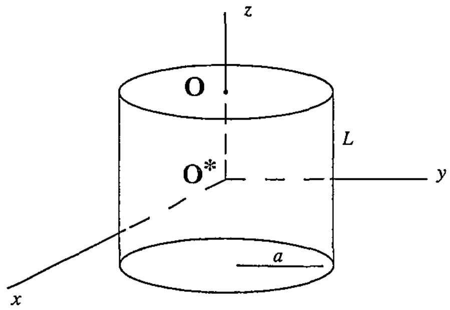
(b) A uniform cylinder of radius a and height L as shown below in its centre of mass coordinate system has a moment of inertia tensor of the form

$$\boldsymbol{I}^* = \begin{pmatrix} \frac{M}{4}(a^2 + \frac{L^2}{3}) & 0 & 0 \\ 0 & \frac{M}{4}(a^2 + \frac{L^2}{3}) & 0 \\ 0 & 0 & \frac{M}{2}a^2 \end{pmatrix}$$

i) Show that the dimensions a and L must be be related as $L = \sqrt{3}a$ if the cylinder is to have exactly the same kinetic energy of rotation when spun with the same angular speed ω about each of the three coordinate axes shown. [4]

ii) Use (without derivation) the parallel axis theorem to obtain the moment of inertia tensor $\boldsymbol{I}$ in the coordinate system whose origin is at the point O shown below, at the centre of the top surface of the cylinder. Explain whether this coordinate system is a principal axis system. [4]

iii) If a pendulum is made by suspending the cylinder from a horizontal axle lying in its top surface and passing through the point O, what would be the frequency of oscillation for small displacements from equilibrium? Evaluate this frequency when L and a are related as in (b) i) above. [8]



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B3

(a) A symmetric top subject to gravity spins with one point on its symmetry axis fixed. Define the Euler angles θ, ϕ and ψ for the top. By means of a sketch, or otherwise, explain what is meant by *precession* and *nutation* of the top. [8]

(b) A rigid body rotates at angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame whose origin O lies at the centre of mass of the body. For a general time dependent vector $\boldsymbol{B}(t)$, *state* (no derivation required) the relation between the time rate of change of $\boldsymbol{B}$ as observed in the inertial frame and as observed in a body-fixed frame of reference that rotates with the rigid body. [4]

(c) Hence, or otherwise, derive the Euler equations of motion for a force- and torque-free rotating rigid body,

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \quad ,$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 \quad ,$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \quad .$$

In what frame of reference do these equations hold? What are the quantities I_1, I_2, I_3? [8]

(d) An asteroid far out in the Kuiper belt for which $I_1 < I_2 < I_3$ spins with a steady angular velocity $\boldsymbol{\omega} = (0, \omega_2, 0)$. We can regard it as both force- and torque-free. A much smaller asteroid collides with it giving it a glancing blow such that immediately after the collision the angular velocity is $\boldsymbol{\omega} = (\epsilon_1, \omega_2, \epsilon_3)$ with $|\epsilon_1| << \omega_2$, $|\epsilon_3| << \omega_2$. Using the Euler equations discuss the time dependence of the angular velocity components $\epsilon_1(t)$, $\epsilon_3(t)$ and determine whether or not the motion subsequent to the collision is stable. [10]

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B4

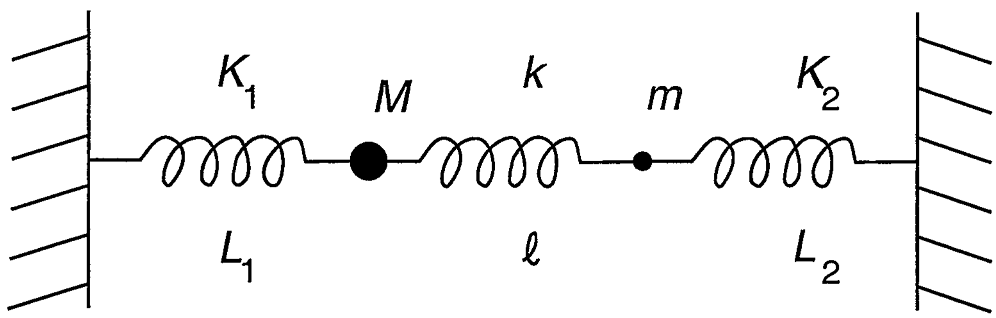
(a) The small oscillation Lagrangian for a system of n degrees of freedom has the form

$$L = \frac{1}{2}\sum_{i,j=1}^{n} m_{ij}^{0}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2}\sum_{i,j=1}^{n} V_{ij}\eta_i\eta_j \quad .$$

Explain briefly what are the quantities η_i, m_{ij}^0 and V_{ij}. Write down the Lagrangian equations of motion and explain briefly what is meant by a normal mode solution of the equations of motion. Give an example of a physical system in which a normal mode analysis is useful. **[10]**

(b) A diatomic molecule consisting of two atoms of masses M and m respectively is trapped between two adjacent crystal surfaces. We can model the forces acting between the two atoms and between the atoms and the wall by massless springs of force constants K_1, k, K_2 and natural lengths L_1, ℓ, L_2 as shown below. Assuming that the atoms move only in the direction normal to the crystal surfaces, write down the Lagrangian for the molecule using appropriate coordinates. Obtain the matrices m_{ij}^0, and V_{ij} for the system. **[10]**

(c) Determine the frequencies and amplitudes of the normal modes of this system if $M = 2m$, $K_1 = 3k$ and $K_2 = k$ **[10]**



END OF EXAM - R. B. JONES

FORMULA SHEET

PLANE POLAR COORDINATES:

$$\boldsymbol{r} = r\,\boldsymbol{e}_r \quad ,$$

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta \quad ,$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad .$$

SPHERICAL POLAR COORDINATES:

$$\boldsymbol{r} = r\,\boldsymbol{e}_r \quad ,$$

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta + r\sin\theta\dot{\phi}\,\boldsymbol{e}_\phi \quad ,$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \quad .$$

CYLINDRICAL POLAR COORDINATES:

$$\boldsymbol{r} = \rho\,\boldsymbol{e}_\rho + z\,\boldsymbol{e}_z \quad ,$$

$$\boldsymbol{v} = \dot{\rho}\,\boldsymbol{e}_\rho + \rho\dot{\theta}\,\boldsymbol{e}_\theta + \dot{z}\,\boldsymbol{e}_z \quad ,$$

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) \quad .$$

VECTOR IDENTITIES

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$