Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-304 Physical Dynamics

Time Allowed: 2 hours 15 minutes

Date: May 7, 2004

Time: 14:30

Instructions: **You should attempt THREE questions, answering the ONE question from section A and TWO questions from section B. The compulsory question in section A is worth 40 %; the questions in section B are worth 30 % each. The mark** *provisionally allocated* **for each sub-section of a question is indicated in square brackets.**

Mathematical formulae: A formula sheet containing mathematical results that may be of help in various questions appears at the end of the examination paper.

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SECTION A: Answer all parts of the **ONE** question in this Section.

1 (a) A system of N point particles with masses m_i, located at positions $\boldsymbol{r}_i$, $i = 1, \ldots, N$, with respect to an inertial frame, moves under the influence of external and internal forces where $\boldsymbol{F}_i^{(e)}$ is the external force on particle i and $\boldsymbol{F}_{ij}$ is the two-body force on particle i due to particle j.

i) Define the position $\boldsymbol{R}$ of the centre of mass of this system. [2]

ii) Explain what is meant by the centre of mass frame of reference for this system giving the transformation equation that takes us from the general inertial frame to the centre of mass frame. [2]

iii) Show how the total linear momentum $\boldsymbol{P}_{tot}$ of the system with respect to the inertial reference frame may be expressed in terms of the time rate of change of $\boldsymbol{R}$. [2]

iv) Derive the result

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + T^* \quad ,$$

where M is the total mass of the system and T^* is the kinetic energy in the centre of mass frame. [6]

v) What conditions on the forces acting on the particles are sufficient for both the total linear and total angular momentum of such a system to be conserved? [2]

(b)

i) Explain what is meant by a set of generalised coordinates for a conservative mechanical system. [2]

ii) Define the Lagrangian for such a system and state Lagrange's equations in terms of the generalised coordinates. [2]

iii) Define what is meant by the generalised momentum p_i and the generalised force Q_i conjugate to the generalised coordinate q_i. [2]

iv) Explain what is meant by a *cyclic* or *ignorable* coordinate. [2]

(c) A planet of mass m moves under the gravitational force of the Sun which can be regarded as a fixed object of mass M.

i) How many degrees of freedom does the planet have? [2]

ii) Using an appropriate set of coordinates, write down the Lagrangian function L for the planet and obtain Lagrange's equations of motion. [8]

iii) Give expressions for all the conserved quantities for the planet and state what symmetries correspond to each of them. [4]

iv) If the planet has a non-zero value of its angular momentum, show that there is a least distance from the Sun that it can attain in its motion. [4]

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SECTION B:

Answer **TWO** questions from this Section.

2

(a) We derived the virial theorem for a conservative system of particles interacting by two-body forces $\boldsymbol{F}_{ij}$ in the form

$$\overline{T} = -\frac{1}{2}\overline{\sum_{i<j} \boldsymbol{F}_{ij} \cdot \boldsymbol{r}_{ij}} \quad .$$

i) Explain what the overbar means. **[2]**

ii) Suppose that the two-body force arises from a two-body potential of the form $v(\boldsymbol{r}) = kr^n$ where k is a constant and n is an integer. Show that the virial theorem can be re-expressed as

$$\overline{T} = \frac{n}{2}\overline{V} \quad ,$$

where V is the total potential energy of the system. **[8]**

iii) Discuss briefly how the virial theorem may be used in the study of the dynamics of galaxies and in the study of the specific heat of crystals whose atoms interact by springlike forces. **[4]**

(b) The Euler equation of motion for an ideal incompressible fluid can be written as

$$\rho\frac{D\boldsymbol{v}}{Dt} = \rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{F} - \boldsymbol{\nabla}p \quad .$$

Without detailed derivations, explain the significance of each term in the equation. What is the difference between the quantities $\frac{D\boldsymbol{v}}{Dt}$ and $\frac{\partial \boldsymbol{v}}{\partial t}$? **[6]**

Starting from the Euler equation derive Bernoulli's theorem, stating clearly any assumptions made. Explain briefly the relevance of Bernoulli's theorem to the surface of a river as it flows over a weir. **[10]**

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3 In an inertial frame of reference a rigid body rotates at angular velocity $\boldsymbol{\omega}$ about a fixed axis.

(a) Taking an origin on the axis of rotation show that the kinetic energy of the rigid body can be expressed as **[4]**

$$T = \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{\omega} \quad .$$

(b) Show also that the angular momentum of the body $\boldsymbol{L}$ is related to its angular velocity $\boldsymbol{\omega}$ by a linear equation of the form $\boldsymbol{L} = \boldsymbol{I}\cdot\boldsymbol{\omega}$ where the moment of inertia tensor $\boldsymbol{I}$ is represented as a three by three matrix. Give explicit expressions for the elements of this matrix. **[8]**

(c) State which elements of $\boldsymbol{I}$ vanish and explain why

i) in the case that the body is reflection symmetric in the $x - y$ plane,

and

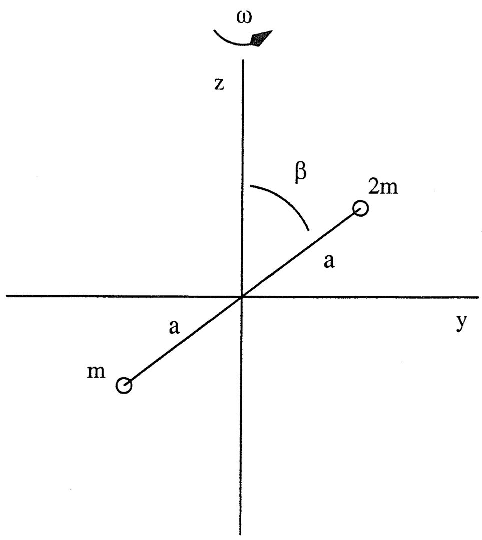
ii) in the case that the $z-$ axis is an axis of rotational symmetry. **[4]**

(d) Two point particles, one of mass m and the other of mass $2m$ are joined by a light rigid rod of length $2a$. This rigid object is shown below in the $y - z$ plane of a Cartesian coordinate system where the rigid connecting rod makes an angle β with the z-axis.

i) Calculate the elements of the moment of inertia tensor for the system of two masses in the coordinate system shown. **[4]**

ii) Define what is meant by a principal axis system. For what values of β is the coordinate system in the diagram a principal axis system? **[4]**

iii) Suppose that the object rotates at constant angular speed ω about the z-axis. At the instant when the object lies in the $y - z$ plane as shown below, evaluate the angular momentum $\boldsymbol{L}$ and the kinetic energy T of the object. Compute the net torque acting on the object at this instant. **[6]**



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(a) A spinning symmetric top can display *precession* and *nutation*. Explain briefly (no mathematical derivations required) what these two phenomena are. Explain briefly how the concept of *precession* applies to the front wheel of a motorcycle turning a corner. **[8]**

(b) A rigid body rotates at angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame whose origin O lies at the centre of mass of the body. For a general time dependent vector $\boldsymbol{B}(t)$, *state* (no derivation required) the relation between the time rate of change of $\boldsymbol{B}$ as observed in the inertial frame and as observed in a body-fixed frame of reference that rotates with the rigid body. **[4]**

(c) Hence, or otherwise, derive the Euler equations of motion for a force- and torque-free rotating rigid body,

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \quad ,$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 \quad ,$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \quad .$$

In what frame of reference do these equations hold? What are the quantities I_1, I_2, I_3? **[8]**

(d) A symmetric asteroid for which $I_1 = I_2 < I_3$ spins with a steady angular velocity $\boldsymbol{\omega} = (0, 0, \omega_3)$. A much smaller asteroid collides with it giving it a glancing blow such that immediately after the collision the angular velocity is $\boldsymbol{\omega} = (\epsilon_1, \epsilon_2, \omega_3)$ with $|\epsilon_1| << \omega_3$, $|\epsilon_2| << \omega_3$. Using the Euler equations discuss the subsequent motion of the large asteroid. Show that the motion is stable in the sense that the angular velocity components $\epsilon_1(t)$, $\epsilon_2(t)$ remain small for all later times. **[10]**

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(a) Briefly compare and contrast the Lagrangian and Hamiltonian formulations of classical mechanics. Explain how the Hamiltonian function is defined and write down the Hamilton equations of motion for a system of n degrees of freedom. **[10]**

(b) Define what is meant by the *action* for a mechanical system and state Hamilton's principle. **[5]**

(i) In special relativity the only time and space dependent quantity, invariant under Lorentz transformations, that we can form for a free point particle of rest mass m_0 is the invariant interval associated with its worldline. Use the invariant interval to formulate the action for the particle. Use Hamilton's principle to deduce the Lagrangian L for the free relativistic particle. **[9]**

(ii) Starting from the Lagrangian for the free relativistic particle obtain its Hamiltonian. **[6]**

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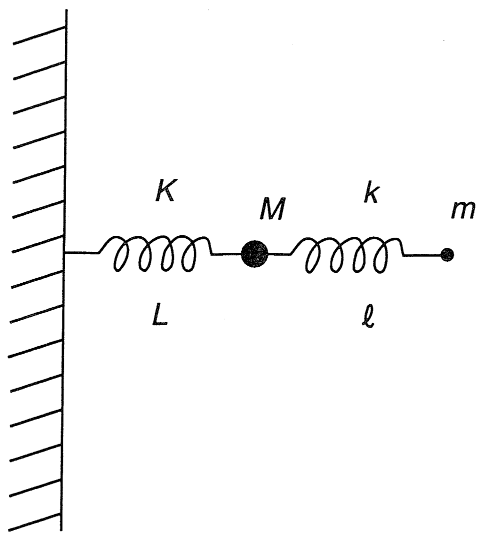
(a) The small oscillation equations of motion for a conservative mechanical system of n degrees of freedom have the form

$$\sum_{j=1}^{n} m_{ij}^{0}\frac{d^2\eta_j}{dt^2} + \sum_{j=1}^{n} V_{ij}\eta_j = 0 \quad ,(i = 1, 2, \ldots, n).$$

Explain briefly what are the quantities η_i, m_{ij}^0 and V_{ij}. What is meant by a *normal mode* of such a system. How many normal modes will such a system possess? Briefly describe one experimental technique used to explore the normal modes of a molecule undergoing small oscillations. **[10]**

(b) A diatomic molecule consisting of two atoms of mass M and m respectively is adsorbed on a solid surface. We can model the forces of attraction in this system (see below) by a spring of force constant K and natural length L holding the atom M to the wall and a second spring of force constant k and natural length ℓ holding atom m to atom M. Assuming that the atoms move only in the direction normal to the crystal surfaces, write down the Lagrangian for the molecule using appropriate coordinates. Obtain the small oscillation equations of motion for the system. **[10]**

(c) Determine the frequencies and amplitudes of the normal modes of this system if $M = 4m$ and $K = 3k$. **[10]**



END OF EXAM - R. B. JONES

FORMULA SHEET

PLANE POLAR COORDINATES:

$$\boldsymbol{r} = r\,\boldsymbol{e}_r \quad ,$$

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta \quad ,$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad .$$

SPHERICAL POLAR COORDINATES:

$$\boldsymbol{r} = r\,\boldsymbol{e}_r \quad ,$$

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta + r\sin\theta\dot{\phi}\,\boldsymbol{e}_\phi \quad ,$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \quad .$$

CYLINDRICAL POLAR COORDINATES:

$$\boldsymbol{r} = \rho\,\boldsymbol{e}_\rho + z\,\boldsymbol{e}_z \quad ,$$

$$\boldsymbol{v} = \dot{\rho}\,\boldsymbol{e}_\rho + \rho\dot{\theta}\,\boldsymbol{e}_\theta + \dot{z}\,\boldsymbol{e}_z \quad ,$$

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) \quad .$$

VECTOR IDENTITIES

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \frac{1}{2}\boldsymbol{\nabla}v^2 + (\boldsymbol{\nabla}\times\boldsymbol{v})\times\boldsymbol{v} \quad .$$