Length Contraction and the Lorentz Transform

Length Contraction

The results of the previous section imply *Length Contraction*, for if Twin B finds that the journey to a planet ten light-years away takes only 17.3 years at 0.5 *c*, it follows that in the spaceship reference frame, for Twin B, the journey length is only 8.66 light-years. (That is, the Earth recedes 8.66 light-years away – of course, in the spaceship's own frame the spaceship hasn't moved at all.)

The *Proper Length* of an object is defined as the length measured by an observer in an inertial frame which is at rest relative to the object. (C.f. definition of proper time.)

We find that ***Length Contraction*** follows straightforwardly in the other examples used in the previous section to demonstrate ***Time Dilation.***

The mu-meson created in the upper atmosphere, with a lifetime of 2.2μs, sees the ground approaching it at almost the speed of light. And the ground arrives at it in less than its lifetime, and therefore starts off less that 2200 feet ($c \sim 1$ ft per nanosecond) or 700 m away.

Consider the burn-marks on the track when the moving train is struck by lightning at both ends, simultaneously according to an observer on the track (Observer O). We have seen that an observer on the train O' considers that the lightning strike at the front of the train occurs *before* the lightning strike at the rear of the train. It follows that he sees the burn-marks on the embankment as *less far apart* than the length of the train.

The proper distance L in a frame S contracts to a distance $L' = \dfrac{L}{\gamma}$ when measured in another frame S' moving at the relative speed v.

Lorentz Transformation

As usual, let S be a frame and S' be a frame moving at v to positive x, with origins coincident at $t = t' = 0$. Then we ask, if an event occurs at (x', y', z', t') in S', what are its coordinates (x, y, z, t) in S? We may immediately note that the origin of S' is at $x = vt$, and the distance from the origin to x' undergoes length contraction when viewed from S. So

$$x = vt + \frac{x'}{\gamma}$$

$$x' = \gamma x - v\gamma t$$

It follows from the Principle of Relativity, with a change in sign for v, that

$$x = \gamma x' + v\gamma t'$$

Equating the two resulting expressions for x,

$$\frac{x'}{\gamma} + vt = \gamma x' + v\gamma t'$$

$$vt = \left(\gamma - \frac{1}{\gamma}\right)x' + v\gamma t'$$

$$t = \frac{1}{v}\left(\gamma - \frac{1}{\gamma}\right)x' + \gamma t'$$

The term in x' simplifies:

$$\gamma - \frac{1}{\gamma} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{1-\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{v^2}{c^2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{v^2}{c^2}\gamma$$

So that $t = \frac{v}{c^2}\gamma x' + \gamma t'$.

It follows from the Principle of Relativity, with a change in sign for v, that

$$t' = \frac{-v}{c^2}\gamma x + \gamma t$$

The full Lorentz forward and back transforms are thus:

$$\boxed{\begin{aligned} x' &= \gamma x - v\gamma t \\ y' &= y \\ z' &= z \\ t' &= \frac{-v}{c^2}\gamma x + \gamma t \end{aligned}} \qquad \boxed{\begin{aligned} x &= \gamma x' + v\gamma t' \\ y &= y' \\ z &= z' \\ t &= \frac{v}{c^2}\gamma x' + \gamma t' \end{aligned}}$$

Intervals between Events:

We have seen that the interval between two events that happen at the same place is the time-like interval, the proper time, and the distance between two events that happen at the same time is the space-like interval, the proper length. What can we say of the interval between two events generally (events at different places and times?

Consider an event at the origin ($x = 0, t = 0$) and an event at (x, t). We are accustomed to using Pythagoras to combine distances on different axes. A spacetime diagram (see Ohanian, *Physics*, 2nd ed., Sec. 41.3) suggests we should try that here.

$$x'^2 = \gamma^2 x^2 + v^2\gamma^2 t^2 - 2v\gamma^2 xt$$

$$t'^2 = \left(\frac{-v\gamma}{c^2}\right)^2 x^2 + \gamma^2 t^2 - 2\frac{v\gamma^2}{c^2}xt$$

Multiplying the second line by c^2, we see that we get rid of the cross term (in xt) by *subtracting*:

$$x'^2 = \gamma^2 x^2 + v^2\gamma^2 t^2 - 2v\gamma^2 xt$$

$$c^2 t'^2 = \frac{v^2\gamma^2}{c^2}x^2 + \gamma^2 c^2 t^2 - 2v\gamma^2 xt$$

$$x'^2 - c^2 t'^2 = \left(\gamma^2 - \frac{v^2\gamma^2}{c^2}\right)x^2 + \left(\frac{v^2\gamma^2}{c^2} - \gamma^2\right)c^2 t^2$$

The coefficients of x^2 and c^2t^2 simplify,

$$\gamma^2 - \frac{v^2\gamma^2}{c^2} = \frac{1}{1-\frac{v^2}{c^2}}\left(1-\frac{v^2}{c^2}\right) = 1$$

so that

$$x'^2 - c^2 t'^2 = x^2 - c^2 t^2$$

or, more generally,

$$s = s' = \sqrt{x'^2 + y'^2 + z'^2 - c^2 t'^2} = \sqrt{x^2 + y^2 + z^2 - c^2 t^2}$$

is the *Lorentz-invariant* interval between two events.

Note that this implies that (x, y, z, ict) is a *four-dimensional space*, in which ict has dimensions of length. Or one may write the space as (x, y, z, ct), with a **Metric (+, +, +, –),** with the sign indicating the contribution to the Pythagorean sum.

Note too that $s = 0$ for events separated by the speed of light, e.g. the creation of a photon at a distant star and its observation on Earth. *For a photon, its journey takes no time and is of zero length – for us, its γ is infinite – so this makes sense.*