EMT Revision Topics

TOPIC 1

Maxwell Eqs + Matter

$$\vec{\nabla}\cdot\vec{D} = \rho \,, \quad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \;; \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 .$$

Def^{n}: $\vec{D} = \epsilon_0\vec{E} + \vec{P}(r)$; $\vec{\nabla}\cdot\vec{P} = -\rho_b$

polarization field

$\vec{\nabla}\cdot\vec{D} = \rho_f$

Typically $\vec{P} \sim \vec{E}$; $\vec{P} = \epsilon_0 \chi_e \vec{E}$; $\Rightarrow \vec{D} = \epsilon_r\epsilon_0\vec{E}$

χ_e electric suscept.

$\vec{H}$ field

analagous to $\vec{D} \leftrightarrow \vec{E}$; $\vec{H} \leftrightarrow \vec{B}$ in magnetic medium.

$$\vec{\nabla}\times\vec{H} = \vec{J}_f \;; \quad \vec{\nabla}\times\vec{B} = (\vec{J}_b + \vec{J}_f)\mu_0$$

$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ magnetization field. $\vec{M} = \chi_m \vec{H}$

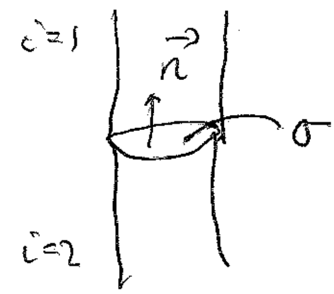
χ_m mag. suscept.

$$\Rightarrow \vec{B} = \mu_0 \mu_r \vec{H}$$

Boundary Conditions on matter interfaces

$(\vec{D}_2 - \vec{D}_1) \cdot \vec{n} = \sigma$ — surface charge density.

$(\vec{B}_2 - \vec{B}_1) \cdot \vec{n} = 0$



$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}$ — surface current

Learn derivation

$\vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0$

Applications: refraction (typical exam quest.)

Related: Plane Polarized Waves — complex notation etc...

Dipole moments: $\vec{d} = \int \vec{P}\, d\vec{V}$; $\vec{m} = \int \vec{M}\, dV$

Expressions for molecular contributions:

$\langle d_{mol} \rangle = \epsilon_0 \gamma_{mol} (\vec{E} + \vec{E}_i)$

Claussius - Mossotti Relation $\gamma_{mol} = \dfrac{3\epsilon/\epsilon_0 - 1}{N\epsilon/\epsilon_0 + 2}$

N = average number molecules / unit vol.

Solving Maxwell's Equations

TOPIC 2

Green's fct^n method:

Introduce magnetic & electric potentials:

$$\vec{\nabla}\times\vec{A} \equiv \vec{B} \quad ; \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

↳ 'wave equations'

$$\nabla^2\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J}$$

$$-\nabla^2\Phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\epsilon_0$$

gauge fixing: e.g. $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$ →

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\rho/\epsilon_0 \quad ; \quad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

other choices .....

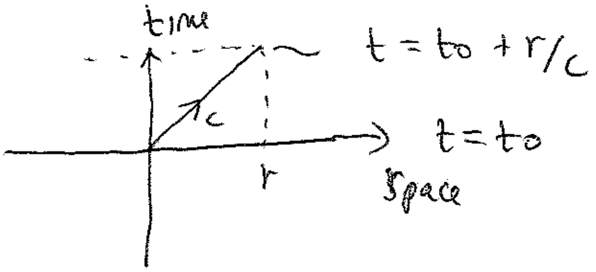
Solutions (should know derivation)

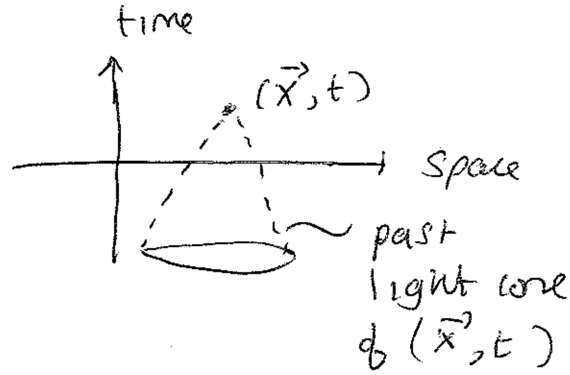
$$\vec{A}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\frac{\mu_0}{4\pi}\int d^3x'\,dt'\,\vec{J}(\vec{x}',t')\,G^+_\omega(\vec{x}-\vec{x}',t')$$

$$G^+_\omega \sim \frac{e^{-i\omega(t'+k|\vec{x}-\vec{x}'|)}}{|\vec{x}-\vec{x}'|}$$ – retarded G.F.

$$\rightsquigarrow \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\vec{J}(\vec{x}', t - r/c)\,\frac{1}{r} \qquad r = |\vec{x}-\vec{x}'|$$

physical interpretation:



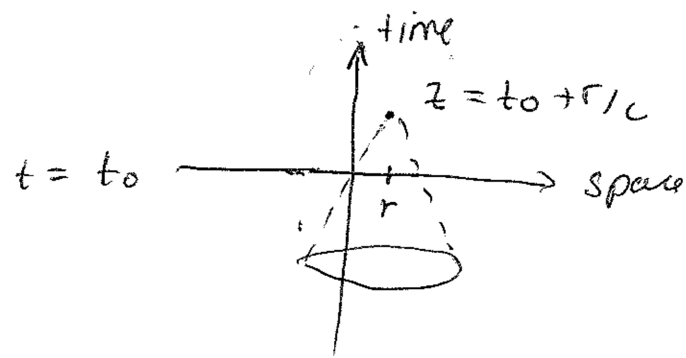


switching on current at time $t = t_0$, takes time $r/c + t_0$ to reach point distance r from origin

- Causality choice of encoded in retarded Green function G^+

Similarly for solution $\Phi(\vec{x}, t)$:-

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} d^3x' \, \rho(\vec{x}', t - r/c) \, \frac{1}{r}$$



Radiation & Multipole Expansions

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \, e^{-i\omega(t-r/c)} \frac{1}{r} \vec{J}(\vec{x}')$$

$r = |\vec{x}-\vec{x}'|$

1) Near zone : $d \ll r \ll \lambda$
2) Int. zone : $d \ll r \sim \lambda$
3) Far zone : $d \ll \lambda \ll r$

↳ radiation region: $\vec{A}_{far}(\vec{x},t) = -\mu_0 \frac{e^{ikr}}{r} i\omega \vec{d}(t)$

plane wave behaviour

$\sim 1/r$ radiation

dipole moment.

For Scalar : $\Phi(r,t) \sim \frac{Q}{4\pi\epsilon_0 r} e^{-i\omega t}$ → Coulomb field (no radiation)

$\vec{E}, \vec{B}$ fields : multipole Expansion

- Far zone - Expand $|\vec{x}'|/r$ powers from $e^{ik|\vec{x}-\vec{x}'|}$

and $\frac{1}{|\vec{x}-\vec{x}'|}$ → multipole moments :

Calc: $\vec{B} = \vec{\nabla}\times\vec{A}_{multipole}$; $\vec{E} = -\vec{\nabla}\Phi_{multipole}$

$$\vec{B}_{dipole} = \frac{k^2}{4\pi\epsilon_0 c} \frac{e^{ikr}}{r} \left[1 - \frac{1}{ikr}\right] \vec{n} \times \vec{d}(t)$$

— $\vec{B}$ field due to Electric dipole.

$\vec{n} \equiv \vec{x}/r$.

$$\vec{E}_{dip} = \underbrace{\frac{k^2}{4\pi\epsilon_0} \frac{e^{ikr}}{r} (\vec{n}\times\vec{d})\times\vec{n}}_{1/r \sim \text{far region}} + \frac{e^{ikr}}{4\pi\epsilon_0 r} \left(\frac{1}{r^2} - \frac{ik}{r}\right) \left(3\vec{n}(\vec{n}\cdot\vec{d}) - \vec{d}\right)$$

— $\vec{E}$ field due to Elec dipole

— revise various limits (e.g far zone)

— Important in far zone: $\vec{E} = c\vec{B}\times\vec{n}$ (c.f. plane wave solutions)

Magnetic dipole:

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = \frac{e^{ikr}}{r} \left[\overset{\text{electric dipole term}}{1} - \underset{\text{mag-dipole term}}{ik\vec{n}\cdot\vec{x}'} + \cdots\right]$$

$\rightarrow$ $\vec{A}_{m.d} = \frac{ik\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{n}\times\vec{M}$ — mag. dipole moment $\sim \frac{1}{2}\int d^3x' \left(\vec{x}'\times\vec{J}(\vec{x}')\right)$

We found: $\vec{A}_{m.d}(\vec{m}) = \frac{i}{kc} \vec{B}_{e.d}(\vec{d} \to \vec{m})$

$$\vec{B}_{m.d} \equiv \vec{\nabla} \times \vec{A}_{m.d} \quad ; \quad \vec{E}_{m.d} = \frac{ic}{k} \vec{\nabla} \times \vec{B}_{m.d}$$

Revise above : also relations: $\vec{B}_{m.d} = \frac{1}{c^2} \vec{E}_{e.d}(\vec{d} \to \vec{m})$

$$\vec{E}_{m.d} = -\vec{B}_{e.d}(\vec{d} \to \vec{m})$$

- duality symmetry between electric / magnetic dipole radiation...

- Quadrupole - usually no questions - quite involved calculations (no time!)

Radiation sources: linear antenna -

notion of $\frac{dP}{d\Omega}$ - power radiated through unit solid angle

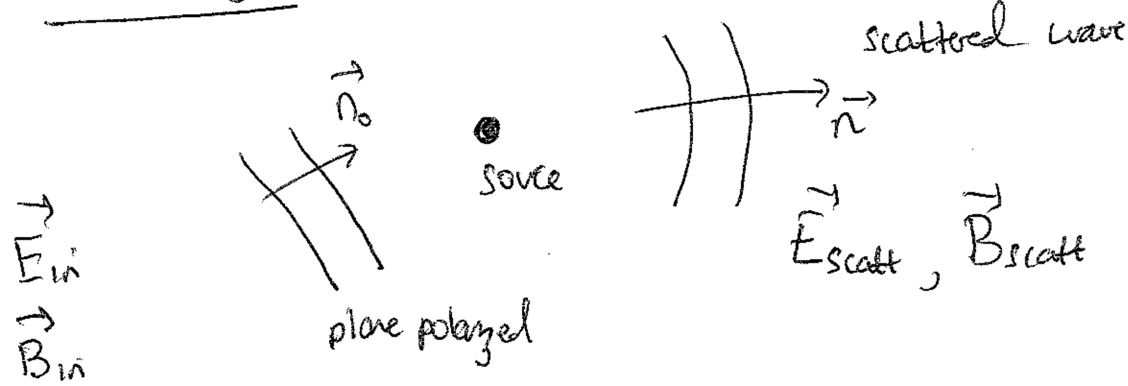
$$P = \int \frac{dP}{d\Omega} d\Omega \quad = \text{total power radiated}$$

$\sim \omega^4$ ($\sim 1/\lambda^4$) typical dipole behaviour.

related to

Poynting vector $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$

Scattering



- We looked at various scattering sources
- Small scatterer localized $\sim d^3$ → calc. far region scattered wave
- random scatterers / coherent scattering.
- Scattering of light off air molecules ↪ Rayleigh scattering.

$$\vec{E}_{in}(x,t) \sim \vec{E}_0(t)\, e^{i\vec{k}_0\cdot\vec{x}}$$

$$\vec{E}_{scatt}(x,t) \sim \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[(\vec{n}\times\vec{d})\times\vec{n} - \frac{\vec{n}\times\vec{m}}{c}\right]$$ – dipole form in far region

Incident flux : $\langle|\vec{S}_{in}|\rangle \sim \frac{1}{2\mu_0 c}|\vec{E}_{in}|^2$ ⇒ time average of Poynting Vector

$$\vec{S}_{in} = \frac{1}{\mu_0}\vec{E}_{in}\times\vec{B}_{in}$$

Similarly scattered flux:

$$\langle |\vec{S}_{sc}| \rangle \sim \frac{1}{2\mu_0 c} |\vec{E}_{sc}|^2$$

$\Rightarrow$ notion of <u>scattering cross-section</u>

$$\frac{d\sigma}{d\Omega}(\vec{n}, \vec{e}\,; \vec{n}_0, \vec{e}_0) \equiv r^2 \frac{|\vec{e}^{\,*} \cdot \vec{E}_{sc}|^2}{|\vec{e}_0 \cdot \vec{E}_{in}|^2}$$

polarization vectors.

typically $\vec{E}_0 = \vec{e}_0 \, e^{-i\omega t} E_0$

$\rightarrow$ Rayleigh scattering $d\sigma/d\Omega \sim 1/\lambda^4$.

<u>Application to scattering of Light</u>

no free charges and currents $\rho_f = \vec{J}_f = 0$ (1st approx).

- source of scattering due to local fluctuations in ϵ.

- We derived $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{D} = \vec{S}(\vec{x},t)$

$$\vec{S}(x,t) = -i\omega/c^2 \, \vec{\nabla}\times\vec{M} - \vec{\nabla}\times(\vec{\nabla}\times\vec{P})$$

$$\begin{cases} \vec{D} = \vec{P} - \epsilon_0 \vec{E} \\ \vec{H} = \vec{M} + \frac{1}{\mu_0}\vec{B} \end{cases}$$

so $\vec{P}, \vec{M}$ <u>provide source of scattering</u>.

$$\vec{D} = \vec{D}_0 + \frac{e^{ikr}}{r}\vec{A}(x) \quad \text{far region}$$

$\uparrow$ incident

Scattering cross-section $\frac{d\sigma}{d\Omega} = \frac{|\vec{e}^{\,*}\cdot\vec{A}|}{|\vec{D}_0|^2}$

Approximation: $\vec{P} = \epsilon_0 \delta\epsilon_r \vec{E}$; $\vec{M} = \frac{\delta\mu_r}{\mu_0}\vec{B}$

– Use Born Approximation $\vec{E} \to \vec{E}_0$; $\vec{B} \to \vec{B}_0$

if $\delta\epsilon_r, \delta\mu_r \ll 1$.

$$\vec{A}(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, e^{-ik\vec{n}\cdot\vec{x}'} \, \vec{S}(x')$$

plug in for $\vec{P}, \vec{M}$ to get $\vec{A}_{born}$.

$$\rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{BORN} = \left(\frac{k^2}{4\pi}\right)^2 \left|\int d^3x \, e^{i\vec{q}\cdot\vec{x}} \left[\vec{e}^{\,*}\cdot\vec{e}_0 \, \delta\epsilon_r\right]\right|^2$$

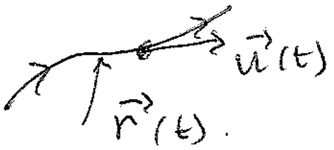
(ignoring $\delta\mu_r$)

$\vec{q} = (\vec{n}_0 - \vec{n})$; $\vec{e}^{\,*}\cdot\vec{n} = 0$

– Looked at examples of using this formulae

e.g. scattering off a dielectric sphere ....

Fields due to charges in Motion



$$J^0 = c\rho = q c\, \delta^{(3)}(\vec{x} - \vec{x}(t))$$

$$\vec{J} = q \vec{u}\, \delta^{(3)}(\vec{x} - \vec{x}(t))$$

Relativistic form: $J^\mu = q\, \gamma_u^{-1}\, U^\mu\, \delta^{(3)}(\vec{x} - \vec{x}(t))$

4 velocity $= (\gamma_u c, \gamma_u \vec{u})$

$\gamma_u = \frac{1}{\sqrt{1-\beta^2}}$; $\vec{\beta} = \vec{u}/c$

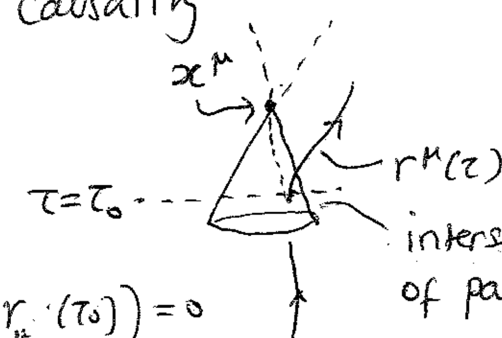
– 4 vector potential obtained by solving Equations of motion

$$A^\mu(x) = \frac{c \mu_0 q}{2\pi} \int d\tau\, U^\mu(\tau)\, \delta\big((x^\mu - r^\mu(\tau))(x_\mu - r_\mu(\tau))\big) \times \Theta(x^0 - r^0)$$

$r^\mu = (ct, \vec{x}(\tau))$

retarded Green's function.

above equation encodes causality



$$(x^\mu - r^\mu(\tau_0))(x_\mu - r_\mu(\tau_0)) = 0$$

intersection of particle world line with past light cone of point x^μ

→ Liénard-Wiechert potentials

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R(1-\vec{\beta}\cdot\vec{n})}\right]_{ret}$$

$$\vec{A} = \frac{\mu_0 q}{4\pi}\left[\frac{\vec{u}}{R(1-\vec{\beta}\cdot\vec{n})}\right]_{ret}$$

ret = evaluated at $t_{ret} = t - R/c$

↳ $\vec{E}, \vec{B}$ fields: $\vec{B} = \frac{1}{c}(\vec{n}\times\vec{E})_{ret}$

$$\vec{E}_{ret} \sim^{\text{far zone}} \frac{q}{4\pi\varepsilon_0}\frac{1}{c}\left[\frac{\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\vec{n})^3 R}\right]_{ret}$$

– looked at simple examples e.g. charged simple Harmonic oscillator ...

↳ Poynting Vector → Power radiated / unit solid angle $\frac{dP}{d\Omega}$.

→ Larmor formula $P = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{c^3}|\dot{\vec{u}}|^2$

→ generalizes to relativistic case – Lienard formula

→ applications to e.g. charged particle in circular motion

→ energy loss due to 'synchrotron' radiation ....

Covariant Electrodynamics

TOPIC 6

– Should revise basic Special Relativity.

Lorentz covariant Maxwells Equations:–

$(\rho, \vec{J}) \rightarrow$ 4 vector current $J^\mu = \rho_0 U^\mu$
↳ 4-velocity

charge conservation $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \rightarrow \partial_\mu J^\mu = 0$

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \longrightarrow \begin{array}{l} \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \\ \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \end{array}$$

Remaining 2 equations are in

$$* \quad \varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \longrightarrow \left\{ \begin{array}{l} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \end{array} \right.$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ & 0 & -B^z & B^y \\ & & 0 & -B^x \\ & & & 0 \end{pmatrix} \qquad F^{\nu\mu} = -F^{\mu\nu}$$

$$* \Rightarrow \quad F_{\rho\sigma} = \partial_{[\rho} A_{\sigma]} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$$

$$A^\mu = (\phi/c, \vec{A})$$

gauge invariance:

$$A_\mu \to A_\mu + \partial_\mu \Lambda(x)$$

$\Lambda(x)$ ← any function of x^μ

Then $\quad F_{\mu\nu} \to F_{\mu\nu}$

$\Rightarrow$ physical $\vec{E}, \vec{B}$ fields are gauge invariant

Stress Tensor

$$\frac{1}{c^2} T^{\mu\nu} = -F^\mu{}_\alpha F^{\nu\alpha} + \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.$$

Conservation of energy / momentum: $\quad \partial_\mu T^{\mu\nu} = 0.$

components

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla}\cdot\vec{P} = 0$$

$\vec{P}$ ← momentum density

$$\frac{\partial P_a}{\partial t} - \partial^b T^M_{ab} = 0$$

T^M_{ab} ← maxwell stress tensor

Including sources: $\quad \partial^\mu T_{\mu\nu} = -F_{\nu\rho} J^\rho$