EMT: Week 8 Lorentz Covariance & Maxwell's Equations

Brief Review of Special Relativity

- Have seen that Maxwell's Equations predict that light propagates as E-M radiation at speed $v = c$ in vacuo - independent of any particular frame of reference.

Einstein used this fact as basis for his Special Theory of Relativity (1905).

- Basic premise is that fundamental laws are covariant (i.e. their form as expressed through equations) remains same in all inertial frames of reference.

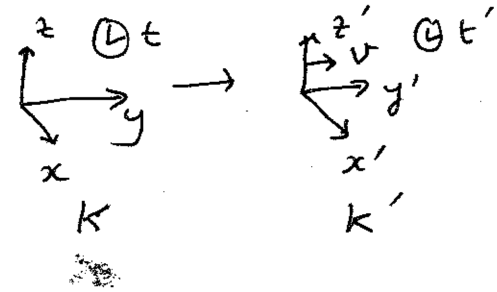
Two frames K, K' are inertial if they differ by uniform rectilinear motion (i.e. no acceleration)

As a particular consequence the speed of light c should be same in all such frames.

This fact implies conditions on co-ordinate transformations relating 2 frames k and k' :-

$$
\begin{aligned}
ct' &= \gamma(ct - \beta x) \\
x' &= \gamma(x - \beta ct) \\
y' &= y \\
z' &= z
\end{aligned}
\qquad (1)
$$

e.g.



where we assumed at $t = t' = 0$, frames co-incide.

An object at rest in k' $(dx' = 0)$ is percieved to be moving with velocity $v = \frac{dx}{dt}$ in k.

$$\Rightarrow \quad dx = \beta c\, dt \,, \quad \frac{dx}{dx} = \beta c \equiv v \qquad \therefore \quad \beta = v/c$$

Then considering wave front of spherical wave of E-M radiation at time t:

$$
\begin{aligned}
x^2 + y^2 + z^2 &= c^2 t^2 \quad \text{in } k \\
x'^2 + y'^2 + z'^2 &= c^2 t'^2 \quad \text{" } k'
\end{aligned}
\qquad (2)
$$

Using linear transformations (1) and $\beta = v/c$

$$(2) \Rightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} .$$

The Lorenz transformation (1) are often called 'Boosts'. (in above case along x direction in k)

Most general boost is when k' is moving along some vector $\vec{v}$ wrt k.

Then equations relating co-ordinates in k, k' are:-

$$ct' = \gamma(ct - \vec{\beta}\cdot\vec{x})$$
$$\vec{x}' = \gamma\vec{x}_{\parallel} + x_{\perp} - \gamma\vec{\beta}ct$$

where $\vec{\beta} = \vec{v}/c$ $\qquad \vec{x}_{\parallel} \equiv \dfrac{(\vec{x}\cdot\vec{\beta})\vec{\beta}}{\beta^2}$ is component of $\vec{x}$ along $\vec{v}$

$\vec{x}_{\perp}$ is component $\perp$ $\vec{v}$.

$\vec{x} = \vec{x}_{\parallel} + \vec{x}_{\perp}$;

Inverse transformations easily found by replacing $\vec{x} \leftrightarrow \vec{x}'$ and $t \leftrightarrow t'$ with $\vec{\beta} \to -\vec{\beta}$.

Four-Vectors

There is an analogy between Lorentz 'Boost' and ordinary rotations in 3d. To see this it's convenient to define $\gamma = \cosh\zeta$ $\quad \therefore \beta = \sqrt{1-\gamma^{-2}} = \tanh\zeta$

Then Boosts look like:-

$$x'^0 = \cosh\zeta\, x^0 - \sinh\zeta\, x^1$$
$$x'^1 = -\sinh\zeta\, x^0 + \cosh\zeta\, x^1$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

where we have defined $x^0 \equiv ct$; $x^1 = x$; $x^2 = y$, $x^3 = z$.

The mixing between x^0, x^1 looks similar to e.g. standard rotation by angle θ about x^3 (z)-axis:

$$x'^1 = \cos\theta\, x^1 + \sin\theta\, x^2 \qquad x'^0 = x^0$$
$$x'^2 = -\sin\theta\, x^1 + \cos\theta\, x^2 \qquad x'^3 = x^3$$

but with $\theta = i\zeta$. Ordinary 3d vectors transform in a natural way under any rotations – to best represent the space-time geometry underlying SR, we introduce notion of a four-vector V^μ (or 'space-time vector') as one which (for the particular boost we have considered above) transforms as:-

$$V^{0\prime} = \cosh\zeta\, V^0 - \sinh\zeta\, V^1$$
$$V^{1\prime} = -\sinh\zeta\, V^0 + \cosh\zeta\, V^1$$
$$V^{2\prime} = V^2$$
$$V^{3\prime} = V^3$$

We may ask what is the generalization of the ordinary scalar product that applies to 3d-vectors, to space-time?

Given 2 four-vectors V^μ, U^μ we can define product:-

$$V^0U^0 - V^1U^1 - V^2U^2 - V^3U^3 \equiv V^0U^0 - \vec{V}\cdot\vec{U}$$

- this is easily checked to be invariant under Lorentz transformations. (Just like scalar product $\vec{A}\cdot\vec{B}$ between 2 3d-vectors $\vec{A}, \vec{B}$ is invariant under ordinary rotations).

As an example of the 'space-time' scalar product between four-vectors we have:

$$ds^2 = (dx^0)^2 - \vec{dx}\cdot\vec{dx} \quad = \text{Lorentz invariant line element.}$$

While different inertial observers at rest in K or K' may both disagree in general on values of (dx^0), $\vec{dx}$ agree on the value of ds^2.

Elementary properties of the Lorentz transformation

(see printed notes) imply 2 important physical

consequences namely:

time dilation

If $d\tau$ is a time interval measured on a clock at rest in k', and dt the time measured on the clock via observer ~~at~~ in k,

$$\boxed{dt = \gamma\, d\tau}$$

$d\tau$ - called 'proper time'
(use inverse L.T at fact that $dx' = 0$.)

$\gamma > 1 \Rightarrow$ 'moving clocks run slow'.

Lorentz-Fitzgerald contraction

- 'moving rods appear contracted along direction of motion'.

$$\boxed{L' = \frac{1}{\gamma} L}$$

$L =$ length of rod at rest in k'
$L' =$ " " measured in k.

- n.b. this is only affects those parts of an object ~~moving~~ ~~is~~ ~~at~~ parallel to the boost direction.

e.g for a boost along x-direction in k, only that part of an object at rest in k' which is parallel to x' axis is contracted.

Four Velocity, Energy & Momentum

Given $dx^\mu = (dx^0, dx^1, dx^2, dx^3)$ is a for-vector

the quantity $\frac{dx^\mu}{d\tau}$ is also a ~~f~~four-Vector because

$d\tau = \frac{1}{c}\sqrt{(dx^0)^2 - d\vec{x}\cdot d\vec{x}}$ is a Lorentz scalar as we have seen.

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \frac{dx^\mu}{dt}\gamma_u \qquad \gamma_u = \frac{1}{\sqrt{1-\vec{u}\cdot\vec{u}/c^2}}$$

$$\vec{u} \equiv \frac{d\vec{x}}{dt}$$

$\therefore$ we can define a four-velocity $U^\mu \equiv \frac{dx^\mu}{d\tau}$

$$U^0 = \gamma_u c \quad ; \quad \vec{U} = \gamma_u \vec{u}$$

Thus any particle with ordinary 3-velocity $\vec{u}$, has a

4-velocity $U^\mu \equiv (\gamma_u c, \gamma_u \vec{u})$ with property.

$$U^2 \equiv U^0 U^0 - \vec{U}\cdot\vec{U} = c^2 \quad \text{(Lorentz invariant)}$$

[τ is time in rest frame of particle - called proper time.

If K' is the rest frame then in K', $ds^2 = c^2 d\tau^2 - \underbrace{d\vec{x}'\cdot d\vec{x}'}_{=0}$

in K, $ds^2 = dx^{0\,2} - d\vec{x}\cdot d\vec{x}$. Hence $d\tau = \frac{1}{c}\sqrt{dx^0 dx^0 - d\vec{x}\cdot d\vec{x}}$]

Non-relativistic case $\vec{p} = m\vec{u}$

$E = \frac{1}{2} m \vec{u}\cdot\vec{u}$.

SR $\rightarrow$ $\vec{p}, E$ replaced by four-momentum

$P^\mu \equiv m_0 U^\mu$ m_0 = rest mass of particle.

4-momentum (P^μ) ; 4-velocity (U^μ)

$\vec{P} = m_0 \vec{U} = m_0 \gamma_u \vec{u}$; $P^0 = m_0 c \gamma_u$

= relativistic 3-momentum. $\equiv E/c$ the relativistic energy.

Clearly $P^0 P^0 - \vec{P}\cdot\vec{P} = m_0^2 (\underbrace{U^0 U^0 - \vec{U}\cdot\vec{U}}_{= c^2})$ - see prev. page.

$\therefore$ $\frac{E^2}{c^2} - P^2 = m_0^2 c^2$ or $\boxed{E^2 - P^2 = m_0^2 c^4}$

- Lorentz invariant relation.

Lorentz Covariant / Contravariant Vectors

We can write Lorentz Boosts in matrix notation:

$$V'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu V^\nu \quad \text{where } \Lambda^\mu{}_\nu = \begin{pmatrix} \cosh z & -\sinh z & 0 & 0 \\ -\sinh z & \cosh z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4 x 4 symmetric matrix

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \quad . \quad \text{Usual to supress } \sum_\mu$$

in matrix multiplication and adopt Einstein summation convention

- where repeated indices are taken to be summed over: -

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu \iff \text{short hand for } V'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu V^\nu .$$

This transformation above implies V^μ is a <u>contravariant</u> vector

(i.e. index μ is superscript and transforms with $\Lambda^\mu{}_\nu$).

We may also define <u>covariant</u> vectors that transform

as e.g. $\frac{\partial f}{\partial x^\mu}$: $\frac{\partial f}{\partial x'^\mu} = \frac{\partial f}{\partial x^\nu}\left(\frac{\partial x^\nu}{\partial x'^\mu}\right) =$

where f any (scalar) function of x^μ.

But $\frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu$ - inverse of $\Lambda^\nu{}_\mu$ above

$\therefore$ Any vector V_μ transforms as :

$$\boxed{V'_\mu \to (\Lambda^{-1})^\nu{}_\mu V_\nu} \quad - \quad \text{covariant vector}$$

Using these transformations, it is possible to see why scalars are Lorentz <u>invariant</u>

e.g. consider 'length' of 4 vector V^μ defined as:-

$V^2 = V^\mu V_\mu$. Then under a L-T.

$$V^{2\prime} = V^{\mu\prime} V_\mu' = \Lambda^\mu{}_\nu V^\nu (\Lambda^{-1})^\rho{}_\mu V_\rho$$

But $\Lambda^\mu{}_\nu (\Lambda^{-1})^\rho{}_\mu = \delta_\nu{}^\rho$

$$\therefore \quad V^{2\prime} = V^\nu V_\rho \delta_\nu{}^\rho = V^2$$

<u>Tensors:</u>

Are generalizations of scalars (no indices), vectors (1-index) to objects with multiple indices....

A general tensor has structure $T^{\overbrace{\alpha\beta\gamma\ldots}^{\text{contravariant indices}}}{}_{\underbrace{\mu\nu\rho\ldots}_{\text{covariant indices}}}$.

Total number of indices = rank of tensor.

Under L.T. tensors transform linearly and homogeneously.

e.g $\quad T_{\mu\nu} \to (\Lambda^{-1})_\mu{}^\rho (\Lambda^{-1})_\nu{}^\sigma T_{\rho\sigma}$

is a rank 2 tensor.

A mixed tensor example is e.g. $D^{\alpha}{}_{\beta}$.

$$D'^{\alpha}{}_{\beta} = \Lambda^{\alpha}{}_{\gamma}\, \Lambda^{-1\,\delta}{}_{\beta}\, D^{\gamma}{}_{\delta} \quad \text{under L.T.}$$

A special tensor is called metric tensor – which is symmetric rank 2, $g_{\mu\nu}$. In Special Relativity this takes on 'Minkowski' form $g_{\mu\nu} = \eta_{\mu\nu}$

$$\eta_{\mu\nu} = \begin{cases} +1 & \mu=\nu=0 \\ -1 & \mu=\nu\neq 0 \\ 0 & \mu\neq\nu. \end{cases}$$

Using it we can define the 'space-time scalar product'

$$U^{\mu} U^{\nu} \eta_{\mu\nu} = (U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2$$

we saw earlier. It has property that it is Lorentz invariant

$$\eta_{\mu\nu} = \Lambda^{\alpha}{}_{\mu}\, \Lambda^{\beta}{}_{\nu}\, \eta_{\alpha\beta} \longrightarrow$$

which follows from $U'^{\mu} U'^{\nu} \eta_{\mu\nu} = U^{\mu} U^{\nu} \eta_{\mu\nu}$

$$= (\Lambda^T \eta \Lambda) = \eta \quad \text{in matrix notation.}$$

where $(\Lambda^T)^{\alpha}{}_{\gamma} = \Lambda_{\gamma}{}^{\alpha}$

* $\eta_{\mu\nu}$ lowers
* index: $V_\mu \equiv \eta_{\mu\nu} V^\nu$
$\eta^{\mu\nu}$ raises: $V^\mu = \eta^{\mu\nu} V_\nu$

Also it's clear that $ds^2 = dx^{\mu} dx^{\nu} \eta_{\mu\nu}$

Maxwell Equations in Lorentz Covariant form

It is a remarkable fact that Maxwell's Equations can in fact be written in a way which shows they are manifestly covariant (i.e. maintain their form) wrt Lorentz transformations (first shown by Lorentz in 1899) - something which Einstein used as a critical argument in his 1905 paper.

We have seen Maxwell Eqs (eg in free space) involve $\vec{E}, \vec{B}$ and sources $\rho, \vec{J}$. We have also seen how $\vec{E}, \vec{B}$ can themselves be written in terms of a vector and scalar potential $\vec{A}, \phi$. ~~resp~~ How can these equations be written in terms of objects like four-vectors, tensors etc.. that we have seen are the natural objects transforming under L.T. ?

First consider sources $\rho, \vec{J}$.

In an inertial frame K, let ρ be a charge density and imagine a charge distribution occupying volume $dxdydz$

$$q = \rho \, dx dy dz .$$

at rest wrt K. Let K' be the inertial frame moving along x-direction with velocity $\vec{v}$ as before.

An observer in K' see's 2 things:-

1) The charge distribution is apparently moving with velocity $\vec{u}' = -\vec{v} = -\vec{\beta}/c$

2) There is consequently a current density $\vec{J}' \propto \vec{u}'$

Given that 1) holds - K' sees Lorentz contraction of

$dx \to \frac{1}{\gamma} dx$ (dy, dz remain unchanged)

$\therefore$ in K', $q' = \rho' dx' dy' dz' = \rho'/\gamma \, dx \, dy \, dz$.

But both observers must measure same charge q (which is a scalar quantity wrt L.T.)

$$\therefore \; q' = q \Rightarrow \rho'/\gamma = \rho \quad \text{or} \quad \boxed{\rho' = \gamma\rho}$$

The apparent current density is then

$$\vec{J}' = \rho' \vec{u}' = (\gamma\rho)(-\vec{\beta}c) = -\gamma\vec{\beta}\rho c$$

This suggests defining a 4-vector current

$J^\mu = (J^0, \vec{J})$ where $J^0 = c\rho$; $\vec{J}$ is current density

We know from previously, a 4-vector transforms as:-

$$J'^\mu = \Lambda^\mu{}_\nu J^\nu$$

$$\Rightarrow \left.\begin{array}{l} c\rho' = \gamma c\rho - \gamma\vec{\beta}\cdot\vec{J} \\ \vec{J}' = \gamma\vec{J}_\parallel + \vec{J}_\perp - \gamma\vec{\beta}\rho c \end{array}\right\} \Rightarrow \text{if } \vec{J} = 0 \quad \rho' = \gamma\rho \, ; \; \vec{J}' = -\gamma\vec{\beta}\rho c \text{ as above!}$$

In fact we can write

$$\boxed{J^\mu = \rho_0 U^\mu}$$

where ρ_0 = charge density in rest frame and

$U^\mu = (\gamma c, \gamma \vec{u})$ is the 4-velocity

- So J^μ transforms as a contravariant vector:

$$J^{\mu'} = \Lambda^\mu{}_\nu J^\nu .$$

charge conservation (continuity equation)

recall ~~...~~ this wave $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$.

Which we can now write in a manifestly Lorentz covariant way :-

$$\boxed{\partial_\mu J^\mu = 0 .}\ !$$

Lorentz force on charged particle.

Recall $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$

for particle moving with velocity $\vec{u}$ wrt some inertial reference frame.

Now $\vec{F} = \frac{d\vec{p}}{dt}$

$\vec{p}$ is part of a 4-vector p^μ as we saw previously.

p^μ transforms nicely under L.T. as a 4-vector.

But Force is $\frac{d\vec{p}}{dt}$ — and so does not transform nicely due to dt factor. However $\frac{d\vec{p}}{d\tau}$ (and by generalization $\frac{dp^\mu}{d\tau}$) does transform as a vector because $d\tau$ is Lorentz invariant since τ, the proper time is.

$$\frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt}\frac{dt}{d\tau} = q\left(\vec{E} + \vec{u}\times\vec{B}\right)\gamma_u$$

$$= q\left(\frac{\vec{E}}{c}U^0 + \vec{U}\times\vec{B}\right) \quad (3) \qquad ; \quad U^\mu = (U^0, \vec{U})$$

Rate of change of energy is

$$\frac{dp^0}{d\tau} = \frac{1}{c}\frac{dt}{d\tau}\frac{d\mathcal{E}}{dt} = \frac{q}{c}\vec{E}\cdot\vec{U} \quad (4)$$

(recall only $\vec{E}$ field does work)
$d\mathcal{E} = q\vec{E}\cdot d\vec{x}$
$\Rightarrow \frac{d\mathcal{E}}{dt} = q\vec{E}\cdot\vec{u}$

We can combine both these equations into a single covariant equation:-

$$\boxed{\frac{d}{d\tau} P^\mu = q F^{\mu\nu} U_\nu} \quad (5)$$

where $U_\nu \equiv \eta_{\nu\mu} U^\mu = (U^0, -\vec{u})$

and the tensor $F^{\mu\nu}$ is:-

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ E^x/c & 0 & -B^z & B^y \\ E^y/c & B^z & 0 & -B^x \\ E^z/c & -B^y & B^x & 0 \end{pmatrix}$$

$\vec{E} = (E^x, E^y, E^z)$

$\vec{B} = (B^x, B^y, B^z)$

e.g. check eqn (4). Take $\mu = 0$ in (5)

$$\frac{d}{d\tau} P^0 = q F^{0i} U_i$$

(note $F^{\mu\nu} = -F^{\mu\nu}$ is an anti-symmetric matrix)

$$= -\frac{q}{c} E^i U_i$$

$$= +q/c \, \vec{E}.\vec{U} \quad ✓$$

since $U_\mu = \eta_{\mu\nu} U^\nu = (U^0, -\vec{U})$

Similarly eqn (3) follows from taking $\mu = i$ in (5).

One might have guessed that $\vec{E}$ and $\vec{B}$ fields would have been part of some 4-vectors in the Lorentz covariant version of Maxwell's Equations - but instead we see they are 'packaged' into a second rank anti-symmetric tensor field $F_{\mu\nu}$ - called Maxwell Field Strength.

What then are the 4 Maxwell Equations in terms of $F_{\mu\nu}$?

Know 3 facts: - 1) Whatever the equations they must be Lorentz covariant - i.e. written in terms of $F_{\mu\nu}$ and it's derivatives. ~~scalar and~~ ~~ea~~

2) Contain at most 1 derivative acting on $F_{\mu\nu}$ - since this is a property of Maxwells Equations in terms of $\vec{E}, \vec{B}$

3) Should contain source terms on r.h.s. - in terms of J^μ

From 1) - 3) we can construct 2 candidates:

$$\partial_\mu F^{\mu\nu} \quad , \quad \varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}$$

Looking at $\partial_\mu F^{\mu\nu}$, consider $\nu=0$ component.

$$\partial_\mu F^{\mu 0} = \partial_0 \cancel{F^{00}}^{\,0} + \partial_i F^{i0} = \frac{1}{c}\vec{\nabla}\cdot\vec{E}.$$

We know one of Maxwell equations is $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

and also that $J^0 = c\rho$. It is clear then that

the covariant equation we seek is:-

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu} \qquad \text{where recall } (\mu_0 \varepsilon_0)^{-1} = c^2$$

- a check is to also take $\nu = i$ components:

$$\partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji}$$

$$= -\frac{1}{c^2}\frac{\partial}{\partial t}E^i + (\vec{\nabla}\times\vec{B})^i = \mu_0 \vec{J}^i$$

which is indeed another of the Maxwell equations!

But this is just 2 out of 4 Maxwell Eqs...

where are other 2 ??

Well consider our second tensor $\left(\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}\right)$

- the missing 2 equations are contained in the additional equation:-

$$\boxed{\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0}$$

Recall that $\varepsilon^{\mu\nu\rho\sigma}$ - Levi-Civita alternating tensor

$$= \begin{cases} \pm 1 & \text{if } \mu\nu\rho\sigma \text{ is } \pm\text{ve permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

i.e. it is cyclically invariant $\epsilon_{\mu\nu\rho\sigma} = \epsilon_{\sigma\mu\nu\rho} = \epsilon_{\rho\sigma\mu\nu}$
$= \epsilon_{\nu\rho\sigma\mu}$

and anti-symmetric on interchanging any pair of indices.

The above equation can be written as:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

is called <u>dual</u> Field strength tensor.

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & -E^z/c & E^y/c \\ B^y & E^z/c & 0 & -E^x/c \\ B^z & -E^y/c & E^x/c & 0 \end{pmatrix} \qquad \text{i.e. } \vec{B} \leftrightarrow \vec{E}/c$$

Now consider e.g. $\nu = 0$

$$\partial_\mu \tilde{F}^{\mu\nu=0} = \partial_0 \cancel{\tilde{F}^{00}}^{\,0} + \partial_i \tilde{F}^{i0} = 0$$

But $\tilde{F}^{i0} = -B^i$ $\therefore$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

taking $\nu = i$ $\quad \partial_\mu \tilde{F}^{\mu i} = 0 = \partial_0 \tilde{F}^{0i} + \partial_j \tilde{F}^{ji} = 0.$

i.e. $\boxed{-\frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \vec{\nabla}\times\vec{E} = 0}$

– precisely our 2 'missing' Maxwell Equations!!

Solving $\partial_\mu \tilde{F}^{\mu\nu} = 0$

$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ can infact be solved:–

$F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$ where A_σ is some 4-vector field

(Ex: check this is true).

The interpretation of A_σ is that they are nothing but the scalar potential and vector potentials $\phi, \vec{A}$ we introduced previously to solve Maxwell's Equations!

E.g. $F_{0i} = E_i/c = \frac{1}{c}\frac{\partial}{\partial t} A_i - \partial_i A_0$.

$$\Rightarrow \quad \vec{E} = -\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi$$ if we identify $A_0 = \Phi/c$

Therefore $A^\mu = (A^0, \vec{A}) = (\Phi/c, \vec{A})$!

So the two potentials $\vec{A}, \Phi$ fit together perfectly into a 4-vector potential, A^μ.

Summary: Covariant Maxwell Equations:-

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\vec{E} = \frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\Phi \quad ; \quad \vec{B} = \vec{\nabla}\times\vec{A}.$$

Note that $F_{\mu\nu}$ is gauge invariant under the gauge transformation:-

$$A_\mu \to A_\mu + \partial_\mu \Lambda \quad ; \quad F_{\mu\nu} \to F_{\mu\nu}$$ [Exercise]

- This is as it should be, because as we have seen, $F_{\mu\nu}$ is made from physical fields $\vec{E}, \vec{B}$ which we know are gauge invariant!

Electromagnetic Stress Tensor

Recall we defined the EM stress tensor in previous lectures:-

$$\frac{1}{\epsilon_0} T^{M}_{ab} \equiv E_a E_b + c B_a B_b - \frac{1}{2}\delta_{ab}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})$$

$$a = 1,2,3$$

Now that we have seen we can express Maxwell's Equations in Lorentz Covariant form, we expect that $T_{ab} \leftrightarrow$ Tensor

$T_{\mu\nu}$ - second rank, symmetric $T_{\nu\mu} = T_{\mu\nu}$.

$(\vec{E}, \vec{B}) \rightarrow$ re-packaged into $F_{\mu\nu}$ - must it

must be possible to express $T_{\mu\nu}$ in terms of $F_{\mu\nu}$.

Answer: $(\epsilon_0 = 1)$ $\boxed{\frac{1}{c^2} T^{\mu\nu} = -F^{\mu}{}_{\alpha} F^{\nu\alpha} + \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}}$

It's easy to check it's components:-

$$T^{00} = \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})$$

$$T^{0b} = -c(\vec{E}\times\vec{B})^b$$

$$-T^{ab} = E_a E_b + c^2 B_a B_b - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{ab}$$

T^{00} - just energy density associated with $\vec{E}, \vec{B}$ (c.f. previous lectures)

T^{0b} - is $(-c) \times \mathcal{P}^b$ - momentum density also previously discussed 'energy current density'

Conservation of energy / Momentum:

In case of no sources ($J^\mu = 0$)

Maxwell Eqs $\rightarrow$ $\partial^\mu F_{\mu\nu} = 0$.

From this it follows that $\boxed{\partial^\mu T_{\mu\nu} = 0}$

— conservation of Stress – Energy.

Look at e.g. $\nu = 0$

$$\partial^\mu T_{\mu 0} = 0 = \left(\partial^0 T_{00} + \partial^i T_{i0}\right)$$

$T_{00} = \mathcal{E}$ = energy density; $T_{i0} = P_i$ – momentum density

① $\dfrac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{P} = 0$ — conservation of Energy

$\nu = i$ ② $\dfrac{\partial}{\partial t} P_a - \partial^b T^M_{ab} = 0$ — conservation of momentum.

Including $J_\nu \neq 0$; $\rightarrow$ $\partial^\mu F_{\mu\nu} = \mu_0 J_\nu$.

$\rightarrow$ $\partial^\mu T_{\mu\nu} = -F_{\nu\rho} J^\rho$

① $\rightarrow$ $\dfrac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{P} = -\vec{J} \cdot \vec{E}$ = w.d by electric currents against $\vec{E}$.

② $\rightarrow$ $\dfrac{\partial P_a}{\partial t} - \partial^b T^M_{ab} = \underbrace{-\rho E_a - (\vec{J} \times \vec{B})_a}_{\text{Lorenz force}}$

Indeed $f^\mu \equiv F^{\mu\nu} J_\nu$
= Lorentz force density
$= \left(\frac{1}{c} \vec{J} \cdot \vec{E},\ \rho\vec{E} + \vec{J} \times \vec{B}\right)$ //