EMT : Scattering II (wk 7)

Reminder of last time: We derived diff^l scattering cross-section in case of incident plane wave scattering off air molecule with induced polarization vector $\vec{P}$ and magnetization $\vec{M}$

$$\vec{D} = \vec{D_0} + \frac{e^{ikr}}{r}\vec{A} \quad ; \quad \vec{A}(\vec{x}) = -\frac{1}{4\pi}\int d^3x' e^{-ik\vec{n}\cdot\vec{x}'}\vec{S}(\vec{x}')$$

$$\vec{S}(x') = -\frac{i\omega}{c^2}\vec{\nabla}\times\vec{M} - \vec{\nabla}\times(\vec{\nabla}\times\vec{P})$$

= "source" term for scattering of incident plane wave $\vec{D_0}$

$$\rightarrow \quad \frac{d\sigma}{d\Omega} = \left|\frac{\vec{e}^*\cdot\vec{A}}{|\vec{D_0}|^2}\right|^2$$

Now since air has $\epsilon \simeq 1$, $\mu \sim 1$; induced $\underline{P}$, $\underline{M}$ expected to be <u>small</u>.

Recall that for weak fields / linear media $\underline{P} \propto \vec{E}$; $\vec{M} \propto \vec{B}$

$$\underline{P} = \vec{D} - \epsilon_0\vec{E} = \epsilon_0(\epsilon_r - 1)\vec{E} \quad ; \quad \vec{M} = (\mu_r - 1)\vec{H}$$

Define $\delta\epsilon_r = (\epsilon_r - 1)$; $\delta\mu_r = (\mu_r - 1)$; $\delta\epsilon_r$, $\delta\mu_r$ both small.

– use them as expansion parameters in 'perturbation theory' approach to finding an expression for scattered field $\vec{A}$.

$$\vec{P} = \epsilon_0 \delta\epsilon_r \vec{E} \; ; \quad \vec{M} = \frac{\delta\mu_r}{\mu_0} \vec{B}$$

To make progress in computing $\vec{A}$ we need to know what $\vec{P}, \vec{M}$ are. Looks like 'circular' problem as they are related to the very $\vec{E}$ and $\vec{B}$ we want to find!

- However if we only need leading order part of $\vec{P}, \vec{M}$ for $\delta\epsilon_r, \delta\mu_r \ll 1$: We can simply take

$$\vec{E} = \vec{E}_0 + O(\delta\epsilon_r) \; ; \quad \vec{B} = \vec{B}_0 + O(\delta\mu_r)$$

where $\vec{E}_0, \vec{B}_0$ are fields associated to <u>incident</u> (unscattered) plane wave! This approximation is called 'Born Approximation' :-

$$\vec{P} \sim \epsilon_0 \delta\epsilon_r \vec{E}_0 = \delta\epsilon_r \vec{D}_0 \; ; \quad \vec{M} \sim \frac{\delta\mu_r}{\mu_0} \vec{B}_0$$
$$\sim \delta\mu_r \, c \, \vec{n}_0 \times \vec{D}_0$$

where we have used $\vec{B}_0 = \vec{n}_0 \times \vec{E}_0 / c$ which holds for incident plane wave.

Integrating by parts:-

$$\vec{A} = \frac{1}{4\pi}\int d^3x'\, e^{-ik\vec{n}\cdot\vec{x}'}\left[\frac{i}{c^2}\omega\,\vec{\nabla}\times\vec{M} + \vec{\nabla}\times(\vec{\nabla}\times\vec{P})\right]'$$

$$= -\frac{k^2}{4\pi}\int d^3x'\, e^{-ik\vec{n}\cdot\vec{x}'}\left[\vec{n}\times(\vec{n}\times\vec{P}') + \vec{n}\times\vec{M}'/c\right].$$

Subing in Born approximation:-

$$\vec{A}_{BORN} = -\frac{k^2}{4\pi}\int d^3x'\, e^{-ik\vec{n}\cdot\vec{x}'}\left[\vec{n}\times(\vec{n}\times\vec{D}_0')\,\delta\epsilon_r' + \vec{n}\times\vec{D}_0'\,\delta\mu_r')\right]$$

$$D_0'(\vec{x}') = D_0\,\vec{e}_0\, e^{ik\vec{n}_0\cdot\vec{x}'} \quad \text{(incident plane wave)}$$

$\vec{e}_0$ ∟ polarization.

$$\rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{BORN} = \left(\frac{k^2}{4\pi}\right)^2 \int d^3x\, e^{i\vec{q}\cdot\vec{x}}\left[\vec{e}^*\cdot\vec{e}_0\,\delta\epsilon_r + (\vec{n}\times\vec{e}^*)\cdot(\vec{n}\times\vec{e}_0)\delta\mu_r\right]\Big|^2$$

$\vec{q} \equiv (\vec{n}_0 - \vec{n})$ & recall $\vec{e}^*\cdot\vec{n} = 0$

[[A check of this formula is provided by considering $\vec{d}$ due to small dielectric sphere - see printed notes

Take use of Clausius-Mossotti relation $\gamma_{mol} = \frac{4}{3}\pi a^3 \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$

with $N = 1$.

$$\vec{d} = \gamma_{mol}\,\vec{E}_0 = 4\pi\epsilon_0 a^3\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)\vec{E}_0 \quad]]$$

Rayleigh's 'Blue Sky' revisited

We now have enough theory based on Born approximation to scattering of light of polarized (magnetized) molecules to try and provide another way of understanding why the sky is blue in colour!

$\delta\epsilon_r = \chi_e \equiv (\epsilon_r - 1)$; we can ignore $\delta\mu_r$ as its $\ll \delta\epsilon_r$ at optical wavelengths.

$\therefore$ the refractive index $n^2 \equiv \epsilon_r \mu_r \sim (\chi_e + 1)$

Previously we found $\left(\frac{d\sigma}{d\Omega}\right)_{BORN}$ due to 1 polarized molecule contributing $\vec{P}$ (take $\vec{M} = 0$ since as we said $\delta\mu_r \ll \delta\epsilon_r$ at optical wavelengths).

Let's consider a contribution from many randomly distributed polarized molecules (having electric dipole moment $\vec{d}_j$)

$$\chi_e(\vec{x}) = \frac{1}{\epsilon_0} \sum_j \gamma_{mol}\, \delta^{(3)}(\vec{x} - \vec{x}_j)$$

where

$$\vec{d}_j = \gamma_{mol} \vec{E}(x_j)$$

$\uparrow$ electric field creating dipole moment of molecule

Substituting for $\delta\epsilon_r(\vec{x}) = \frac{1}{\epsilon_0}\sum_j \gamma_{mol}\,\delta^{(3)}(\vec{x}-\vec{x}_j)$

we find:

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{k^2}{4\pi}\right)^2 \left| \int d^3x\, e^{i\vec{q}\cdot\vec{x}}\; \vec{e}^{\,*}\cdot\vec{e}_0 \frac{1}{\epsilon_0}\sum_j \gamma_{mol}\,\delta^{(3)}(\vec{x}-\vec{x}_j)\right|^2$$

$$= \left(\frac{k^2}{4\pi\epsilon_0}\right)^2 \left|\vec{e}^{\,*}\cdot\vec{e}_0\right|^2 \gamma_{mol}^2\, N$$

where used that $\left|\sum_j e^{iq\cdot\vec{x}_j}\right|^2 = N$ for random distribution of molecules (see last time) 'incoherent scattering'

We have simplified things by taking each of the N molecules to have same polarizability, γ_{mol}.

γ_{mol} is defined as molecular polarizability. We need to relate this to bulk property of our medium (e.g <u>bulk</u> polarizability). This precise relation we derived earlier via Clausius-Mossotti relation:

$$\gamma_{mol} = \frac{3\epsilon_0\,(\epsilon_r - 1)}{\rho_N\,(\epsilon_r + 2)}$$

$\rho_N = N/V =$ number density of molecules.

$\Rightarrow$ in terms of refractive index n :-

$$\frac{\gamma_{mol}}{4\pi\epsilon_0} = \frac{3}{4\pi\rho_N}\left(\frac{n^2-1}{n^2+2}\right) \qquad \left(\begin{array}{c} n^2 = \epsilon_r \\ \text{if } \mu_r \equiv 1 \end{array}\right)$$

$$\therefore \left(\frac{d\sigma}{d\Omega}\right)_{mol} = k^4\,|\vec{\epsilon}^{\,*}\cdot\vec{\epsilon}_0|^2\left(\frac{3}{4\pi\rho_N}\left(\frac{n^2-1}{n^2+2}\right)\right)^2$$

$$/\!/$$

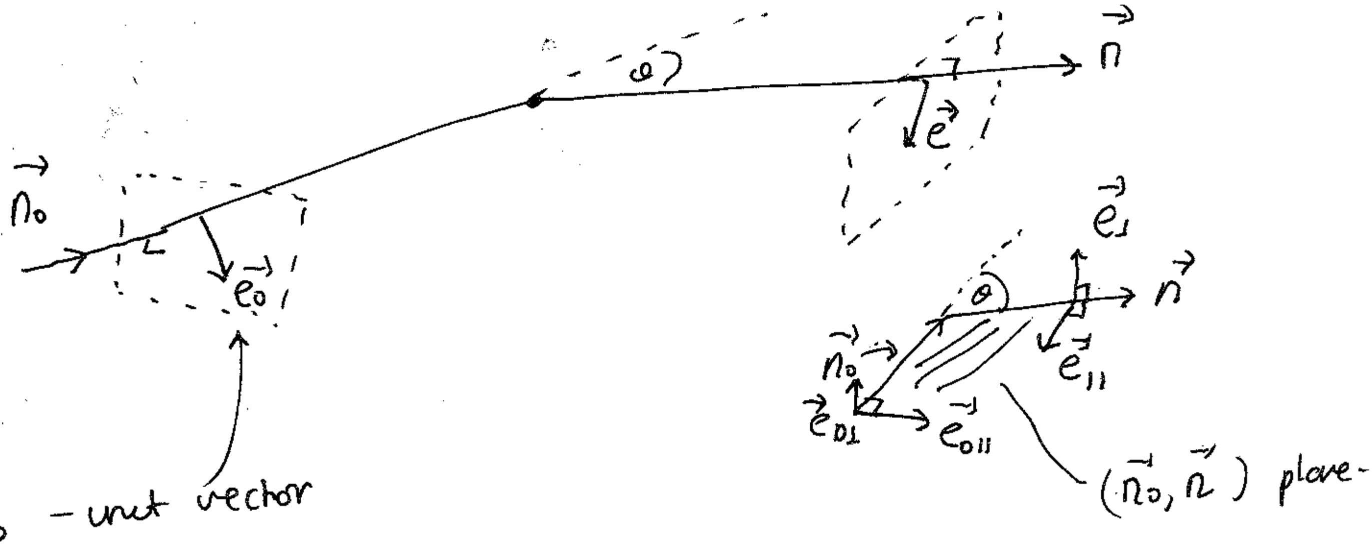
$$\frac{1}{N}\left(\frac{d\sigma}{d\Omega}\right) \sim \boxed{k^4\,|\vec{e}^{\,*}\cdot\vec{e}_0|^2\left(\frac{1}{2\pi\rho_N}\right)^2(n-1)^2}$$

((where $(n-1) \ll 1$ – taylor expand in $(n-1)$: $\left\langle \begin{array}{l} (n^2-1) = \overset{=2}{(n+1)}(n-1) \\ n^2+2 \simeq 3 \end{array}\right.$))

Recall polarization of incident / scattered wave is encoded in the two (generally complex) unit vector $\vec{e}_0$, $\vec{e}$

Incident sunlight hitting atmosphere is unpolarized

– so we should average over the 2 possible polarizations of the incoming / incident wave



decompose: $\vec{e}_0 = \sin\alpha\, \vec{e}_{0\parallel} + \cos\alpha\, \vec{e}_{0\perp}$ $\qquad$ $\vec{e} = \vec{e}_\parallel + \vec{e}_\perp$

Averaging over 2 polarization states of incident wave $\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} d\alpha \left(\frac{d\sigma}{d\Omega}\right)_{mol}$

$$|\vec{e}^{\,*}\cdot\vec{e}_0|^2 = |\vec{e}^{\,*}\cdot\vec{e}_{0\parallel}\sin\alpha + \vec{e}^{\,*}\cdot\vec{e}_{0\perp}\cos\alpha|^2$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, |\vec{e}^{\,*}\cdot\vec{e}_0|^2 = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \Big(\sin^2\alpha\, |\vec{e}^{\,*}\cdot\vec{e}_{0\parallel}|^2 + \cos^2\alpha\, |\vec{e}^{\,*}\cdot\vec{e}_{0\perp}|^2 + \cancel{\tfrac{1}{2}}\sin 2\alpha\, (\vec{e}^{\,*}\cdot\vec{e}_{0\parallel})(\vec{e}\cdot\vec{e}_{0\parallel}^{\,*}) \Big)$$

$$= \frac{1}{2}|\vec{e}^{\,*}\cdot\vec{e}_{0\parallel}|^2 + \frac{1}{2}|\vec{e}^{\,*}\cdot\vec{e}_{0\perp}|^2$$

Taking $\vec{e}^{\,*}$ to be $\parallel$ to $\vec{n}_0$-$\vec{n}$ plane; $(\vec{e}^{\,*}\cdot\vec{e}_{0\parallel}) = \cos\theta$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\parallel} = \frac{1}{2} k^4 \frac{1}{(2\pi\rho_N)^2}(n-1)^2\cos^2\theta$$

= diff^l scatt. cross-section for scattered waves have polarization $\parallel$ to $(\vec{n}_0, \vec{n})$ plane.

Similarly if we take $\vec{e}$ to be ⊥r to $(\vec{n}_0, \vec{n})$ plane; since $|\vec{e}^{\,*} \cdot \vec{e}_{0s}|^2 = 1$.

$$\left(\frac{d\sigma}{d\Omega}\right)_\perp = \frac{1}{2} k^4 \frac{1}{(2\pi\rho_N)^2}(n-1)^2.$$

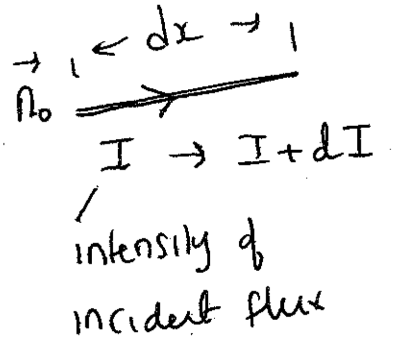
∴ total diff^l cross-section:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_T = \left(\frac{d\sigma}{d\Omega}\right)_\parallel + \left(\frac{d\sigma}{d\Omega}\right)_\perp = \frac{1}{2} k^4 \frac{1}{(2\pi\rho_N)^2}(n-1)^2(1+\cos^2\theta)}$$

from which the total scattering cross-section / molecule :-

$$\boxed{\sigma_{mol} = \frac{2}{3\pi}\frac{k^4}{\rho_N^2}(n-1)^2} \qquad \left(\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\,(1+\cos^2\theta) \to 16\pi/3\right)$$

– power scattered <u>from</u> incident flux <u>per molecule</u> ,



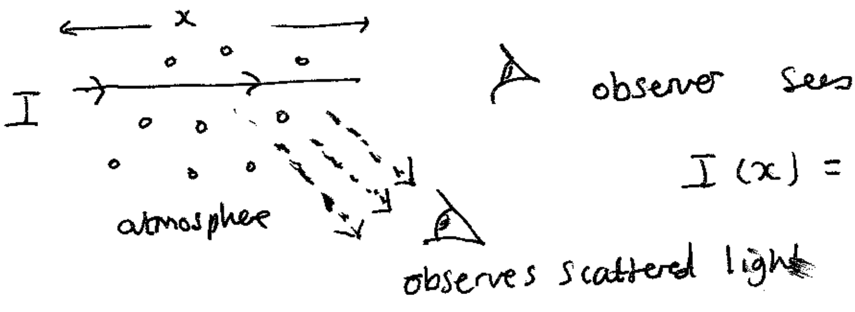
molecules / unit volume

$$dI = -I \rho_N \sigma_{mol}\, dx.$$

⇒ incident beam travels through thickness x of atmosphere

$$I(x) = I_0 e^{-\alpha x}$$

$\alpha \equiv$ attenuation coeff $= \sigma_{mol}\rho_N = \frac{2}{3\pi} k^4/\rho_N (n-1)^2$



$$I(x) = I_0 e^{-\alpha x}.$$

- since α has λ dep such that bluer (shorter) wavelengths have larger values of α than red light

observer looking directly at incident white light from Sun will observe redish appearance because shorter wavelengths have scattered in directions $\vec{n}$ away from initial direction $\vec{n}_0$.

- Effect more noticable at sunset / sunrise when light passes through thickest length of atmosphere than e.g. when directly overhead.

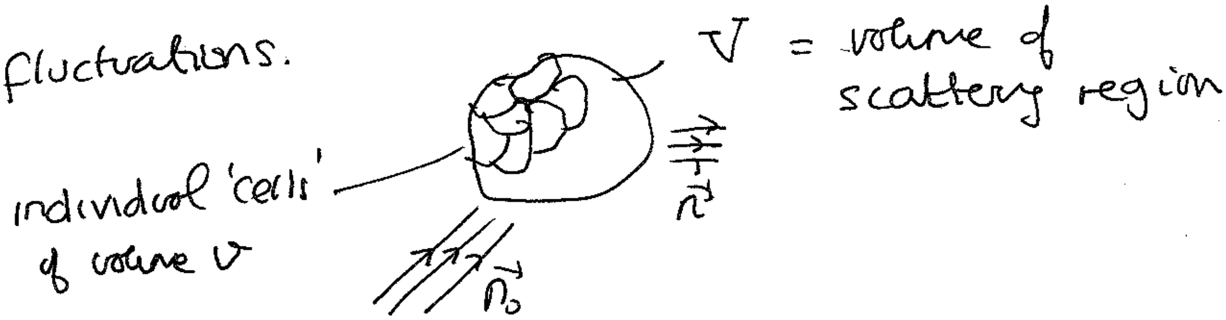
Meanwhile viewing sky away from Sun, - we are seeing the <u>scattered</u> waves - which are predominantly in bluer part of spectrum due to k^4 factor in σ (or $\frac{d\sigma}{d\Omega}$).

[n.b. there is also a dependence on λ via the refractive index n which we have ignored but which also contributes]

Critical Opalescence

Our previous discussions about scattering are fine if medium is basically uniform and ρ_N not too large. But for dense medium we expect density fluctuations might be the origin of scattering more than individual molecules in a uniform medium.

Let's consider a model which can include these density fluctuations.



average number of molecules in a cell $\sim \rho_N v$

where ρ_N = average # molecules / unit volume ← macroscopic

$(v)^{1/3} \ll \lambda$ (so linear size of cell $\ll \lambda$)

- but still take $v\rho_N \gg 1$ (still large # molecules in each cell)

A density fluctuation in cell 'j' ∃ fluctuation in number of molecules inside it, compared to $\rho_N v$.

$\Delta_j \equiv$ fluctuation in number of molecules in j^{th} cell

i.e. $(\rho_N v + \Delta_j)$ = number of molecules in j^{th} cell. $\Rightarrow \delta\rho_j = (\Delta_j / v)$

How does this effect the relative permittivity of j^{th} cell? Well have seen $\epsilon_r = \epsilon_r(\rho_N)$

$$\therefore \; \delta\epsilon_{r_j} = \frac{\partial \epsilon_r}{\partial \rho_N}\,\delta\rho_N = \frac{\partial \epsilon_r}{\partial \rho_N}\,\frac{\Delta_j}{v}$$

From Clausius-Mosotti relation we learn:

$$\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{4\pi\rho_N}{3}\,\frac{\gamma_{mol}}{4\pi\epsilon_0} \Rightarrow \frac{\partial \epsilon_r}{\partial \rho_N} = \frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3\rho_N}$$

$$\rightarrow \quad \delta\epsilon_{r_j} = \frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3\rho_N}\,\frac{\Delta_j}{v} \qquad \left[\frac{\Delta_j / v}{\rho_N} \sim \frac{\delta\rho}{\rho} \ll 1 \right]$$

Recall (ignoring $\delta\mu_r$ so take $\mu_r = 1$)

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{k^2}{4\pi}\right)^2 \left| \int_V d^3x\, e^{i\vec{q}\cdot\vec{x}}\, \vec{e}^{\,*} \cdot \vec{e}_0\, \delta\epsilon_r \right|^2$$

– So now need to include $\delta\epsilon_r$ from each cell $\left(\# = \frac{V}{v}\right)$

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{k^2}{4\pi}\right)^2 \left| \sum_{j=1}^{j=V/v} v\, \vec{e}^{\,*} \cdot \vec{e}_0\, \frac{(\epsilon_r - 1)(\epsilon_r + 2)}{3\rho_N v}\, \Delta_j \right|^2$$

(taken $e^{i\vec{q}\cdot\vec{x}} \simeq 1$ because $(v)^{1/3} \ll \lambda$, so inside cells $|\vec{q}\cdot\vec{x}| \ll 1$)

$$\to \left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{k^2}{4\pi}\right)^2 |\vec{e}^* \cdot \vec{e}_0|^2 \left[\frac{(\epsilon_r-1)(\epsilon_r+2)}{3\rho_N}\right]^2 |\sum_j (\Delta_j)|^2$$

Now averaging over initial polarizations $\vec{e}_0$, as before gives factor $(\frac{1}{2})$. $\int d\theta \, d\phi \sin\theta$ and averaging over final polarization $\vec{e}$ gives $\frac{16\pi}{3}$ $\Rightarrow$ overall $\frac{8\pi}{3}$:

$$\sigma = \left(\frac{k^2}{4\pi}\right)^2 \frac{8\pi}{3}\left[\frac{(\epsilon_r-1)(\epsilon_r+2)}{3\rho_N}\right]^2 |\sum_j \Delta_j|^2$$

Scattering cross-section from $\rho_N V$ molecules inside region V. $\therefore \sigma_{mol} = \sigma/\rho_N V$

Attenuation coefficient $\alpha = \sigma_{mol} \times \rho_N = \left(\frac{k^2}{4\pi}\right)^2 \frac{8\pi}{3}\left(\frac{(\epsilon_r-1)(\epsilon_r+2)}{3\rho_N}\right)^2 \frac{(\sum_j \Delta_j)^2}{V}$

$(\to \; I(x) = I_0 e^{-\alpha x}$ as before$)$

$|\sum_j \Delta_j|^2$ - is the square of the sum of fluctuation in number of molecules taken over all cells in V.

- This can be related to macroscopic properties of medium using statistical mechanical result:

$$\Delta_V^2/\rho_N V = \rho_N k_B T \beta_T \quad ; \quad \beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

= isothermal compressibility.

where $|\sum_j \Delta_j|^2 = \Delta_V^2$ = square of fluctuation in # particles in vol V.

k_B = Boltzmann Constant. , T temperature.

$$\Rightarrow \boxed{\alpha = k^4 \frac{2}{3\pi\rho_N}\left[\frac{(\epsilon_r-1)(\epsilon_r+2)}{6}\right]^2 \rho_N k_B T \beta_T}$$

Called Einstein - Smoluchowski Equation.

For ideal gas $\rho_N k_B T \beta_T = 1$ and also for $\delta\epsilon_r \ll 1$ $\frac{(\epsilon_r-1)(\epsilon_r+2)}{6} \sim (n-1)$

– recover $\alpha = k^4 \frac{2}{3\pi\rho_N}(n-1)^2$ result previously.

However something novel happens at certain critical temp. T_c when $\lim_{T\to T_c} \beta_T \to \infty$ $\therefore \alpha \to \infty$!

-> phenomenon of 'critical opalescence'

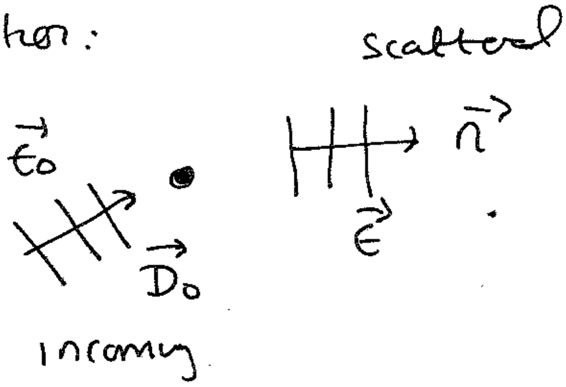
$I(x) \to 0$ (since $e^{-\alpha x} \to 0$) – so all incident light scattered! This phenomenon can be observed in certain mixtures of liquids – at $T \to T_c$, previously transparent solution suddenly becomes opaque as all light is scattered

– however more careful analysis is needed because e.g. cell sizes grow and violate $v^{1/3} \ll \lambda$.

$v^{1/3}$ ~ correlation length

Optical Theorem

This, like the kramers-kronig relation earlier, is a result that basically follows from causality (via Green's functions) - and relates total cross-section to scattering in forward direction:



At large distance from source of scattering:

$$\vec{D} = \vec{D}_0 + \frac{e^{ikr}}{r}\vec{A}$$

and we have shown
$$\frac{d\sigma}{d\Omega} = \frac{|\vec{e}^{*}\cdot\vec{A}|^2}{|\vec{D}_0|^2}$$

where
$$\vec{A} = +\frac{1}{4\pi}\int d^3x' \, e^{-i\vec{k}\cdot\vec{x}'}\,\vec{S}(\vec{x}')$$

$$\vec{S} = -\frac{i\omega}{c^2}\vec{\nabla}'\times\vec{M} - \vec{\nabla}'\times\vec{\nabla}'\times\vec{P}$$

Using various vector identities including Green's Theorem (Stokes theorem in 2d) + Divergence theorem [see book by charap p147 for explicit derivation] one may re-express $\vec{A}$ as:-

$$\vec{A} = \frac{i}{4\pi}\int_{S'} e^{-i\vec{k}\cdot\vec{x}} \left[\omega(\vec{n}\times\vec{B}_s) + \vec{k}\times(\vec{n}\times\vec{E}_s) \right] dS$$

where $\vec{k} = k\vec{n}$; S' is any surface enclosing scatterers.

$\vec{E}_s, \vec{B}_s$ are fields corresponding to <u>scattered</u> wave:

$$\vec{A} = \vec{A}_0 + \boxed{\vec{A}_{sc}} \longrightarrow \left(\begin{array}{ll} \vec{B}_{sc} = \vec{\nabla}\times\vec{A}_{sc} & \\ \vec{E}_s = i\omega\,\vec{A}_{sc} & \text{via Maxwell Eqns:} \end{array} \right)$$

Then we can define a scattering <u>amplitude</u>

$F(\vec{k},\vec{e};\vec{k}_0,\vec{e}_0)$ for scattering incoming wave with polarization $\vec{e}_0$ and wave vector $\vec{k}_0$ to final wave vector $\vec{k}$ and polarization $\vec{e}$:

$$F(\vec{k},\vec{e};\vec{k}_0,\vec{e}_0) \equiv \frac{\vec{e}^{*}\cdot\vec{A}(\vec{k})}{|\vec{D}_0|}$$

So that $\left(\frac{d\sigma}{d\Omega}\right) = |F|^2$ (so F is an 'amplitude') recall that $\vec{A} = \vec{A_0} + \vec{A}_{sc}$;

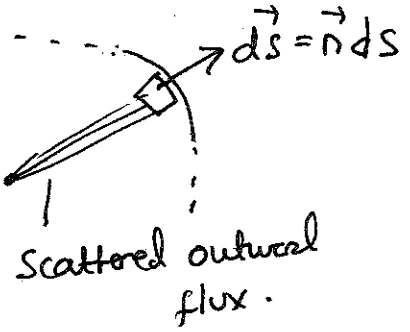
$$\rightarrow \quad \vec{E} = \vec{E_0} + \vec{E}_s \; ; \; \vec{B} = \vec{B_0} + \vec{B}_s$$

Now in general scatter will not just scatter but also absorb radiation incident on it.

The time averaged Power that is scattered we found before:-

$$\vec{P}_{scatt} = \frac{1}{2\mu_0} \int R_e [(\vec{E}_s \times \vec{B}_s^*) \cdot \vec{n}] \, ds$$

from $\frac{1}{T} \int_0^T dt \left[R_e(\vec{E}_s(\vec{x}, t)) \times Re(\vec{B}_s(\vec{x}, t)) \right] \cdot \vec{n} \, ds$



Calculate absorbed power by scatterer;—

[illegible] inward [illegible]

$$P_{absorbed} = \overset{P_{incident}(\vec{n}) \;\leftarrow\; \text{not in notes!!}}{\wedge} - \frac{1}{2\mu_0} \int R \, [(\vec{E} \times \vec{B}^*) \cdot \vec{n}] \, ds$$

n.b no subscript s!

$$P_{incident} = \frac{1}{2\mu_0} \int R(\vec{E_0} \times \vec{B_0}^*) \cdot \vec{n}$$

= component of incident power along $\vec{n}$.

Hence total power <u>removed</u> from incident wave (either by absorption or scattering)

$$\underset{\qquad\qquad\qquad\qquad\qquad}{} P_{incident}(\vec{n})$$

$$P = +\frac{1}{2\mu_0}\int R\left[\left(\vec{E}_s \times \vec{B}_s^*\right)\cdot\vec{n} - Re\left[\vec{E} \times \vec{B}^*\right]\cdot\vec{n}\right]$$

use $\vec{E} = \vec{E}_0 + \vec{E}_s$; $\vec{B} = \vec{B}_0 + \vec{B}_s$

$$= \frac{1}{2\mu_0}\int\left\{R\left(\vec{E}_0 \times \vec{B}_0^*\right)\cdot\vec{n} - \frac{1}{2\mu_0}Re\left[\vec{E}_0 \times \vec{B}_0^*\right]\cdot\vec{n}\right\}ds$$

$$+\frac{1}{2\mu_0}\int R\left[\left(E_s \times \vec{B}_0^* + \vec{E}_0^* \times \vec{B}_s^*\right)\cdot\vec{n}\right]dS$$

$$= \frac{1}{2\mu_0}\int Re\left[\left(\vec{E}_s \times \vec{B}_0^*\right) + \left(\vec{E}_0 \times \vec{B}_s^*\right)\right]\cdot\vec{n}\,ds$$

Taking $\vec{E}_0 = E_0\vec{e}_0\, e^{i\vec{k}_0\cdot\vec{x}}$; $\vec{B}_0 = \frac{c}{k}\vec{k}_0 \times \vec{E}_0$

$(\vec{k}_0 = k\,\vec{n}_0)$

$$P = \frac{1}{2\mu_0}R\left\{E_0^*\int e^{-i\vec{k}_0\cdot\vec{x}}\,\vec{e}_0^*\cdot\left[\vec{n}\times\vec{B}_s + \frac{\vec{k}_0\times(\vec{n}\times\vec{E}_s)}{ck}\right]\right\}dS$$

Now <u>total</u> cross-section (not just scattering cross-section)

$$\sigma_{TOT} \equiv P/\underset{Flux}{incident} = \frac{P}{\epsilon_0|E_0|^2 c/2}$$

$$\sigma_{TOT} = \mathrm{Re}\left\{ c E_0^{-1} \int e^{-i\vec{k}_0\cdot\vec{x}}\, \vec{e}_0^{\,*}\cdot\left[\vec{n}\times\vec{B}_s + \vec{k}_0\times(\vec{n}\times\vec{E}_s)/ck\right] ds \right\}$$

However going back to our definition of scattering amplitude $F(\vec{e},\vec{k};\vec{e}_0,\vec{k}_0)$

we see that :-

$$\boxed{\sigma_{TOT} = \frac{4\pi}{k}\,\mathrm{Im}\,F(\text{forward, elastic})}$$

$$F(\text{forward, elastic}) \equiv F(\vec{k}_0,\vec{e}_0;\vec{k}_0,\vec{e}_0)$$

= amplitude of forward scattered wave $(\vec{n}=\vec{n}_0)$ and no change in polarization $(\vec{e}=\vec{e}_0)$

This relation has far reaching implications beyond just scattering / absorption of light - also true in QFT when discussing scattering / absorption or 'annihilation processes in many particle collisions ↔ concept of <u>unitarity</u>.

in-elastic scattering