Radiation & Multipole Expansions

In these lectures we will make use of last weeks solutions to wave eqn for $\vec{A}$, φ using Green's functions, to calculate radiation produced by various sources.

Multipole Expansion

Let's consider oscillating (in time) sources:-

$$\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}$$
$$\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$$

We will work in Lorenz gauge (like last time)

where we found solution to wave eqn for $\vec{A}$:-

$$\vec{A}(\vec{x},t) = A(\vec{x})e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3x' \, e^{-i\omega(t-r/c)} \frac{1}{r} \vec{J}(\vec{x}')$$

where recall $r = |\vec{x}-\vec{x}'|$. $\therefore$ time-depencence drops out :-

$$\vec{A}(x) = \frac{\mu_0}{4\pi}\int d^3x' \, \vec{J}(x') \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

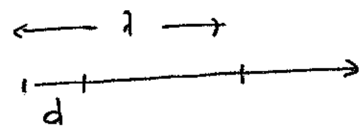
with $k = \frac{\omega}{c} = 2\pi/\lambda$ (assuming free space).

$\int$ on rhs is hard to do - so we will look at some approximations. First lets make (reasonable) assumption that the source $\vec{J}(x')$ is localized within some region of 3d-space of typical size d^3, and that $\lambda \gg d$.

We can then imagine 3 regions over which r varies:-

1) Near zone: $d \ll r \ll \lambda$ (r 'closer' to source)

2) Intermediate Zone $d \ll r \sim \lambda$



3) Far Zone $d \ll \lambda \ll r$

1) $kr = \dfrac{2\pi r}{\lambda} \ll 1$ - so can expand out exponential

$$e^{ikr} \sim 1 + O(kr)$$

$$\vec{A}(x) = \int d^3x' \frac{\vec{J}(x')}{|\vec{x}' - \vec{x}|}$$

lost wave like behaviour in this approx (no e^{ikr})

$\sim$ quasi stationary - osc at freq ω.

3) Far zone; $kr \gg 1$,

Let's define $\vec{x} = r\vec{n}$ $(r = |\vec{x}|)$

$$|\vec{x}-\vec{x}'| = \left[(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')\right]^{\frac{1}{2}}$$

$$= \left(r^2 - 2r\vec{n}\cdot\vec{x}' + \vec{x}'\cdot\vec{x}'\right)^{\frac{1}{2}}$$

$$= r\left(1 - 2\frac{\vec{n}\cdot\vec{x}'}{r} + \frac{\vec{x}'\cdot\vec{x}'}{r^2}\right)^{\frac{1}{2}}$$

$$= r\left(1 - \frac{\vec{n}\cdot\vec{x}'}{r} + \dots\right) = r - \vec{n}\cdot\vec{x}' + \underbrace{\dots}_{O\left(\frac{|\vec{x}'|}{r}\right)}$$

$\Rightarrow$

$$k|\vec{x}-\vec{x}'| = kr - k\vec{n}\cdot\vec{x}' + O\left(\frac{k|\vec{x}'|}{r}\right).$$

$$\therefore \quad e^{ik|\vec{x}-\vec{x}'|} \approx e^{ikr} + \dots$$

Keeping leading term only;

$$\vec{A} \approx \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}(x')$$

($I.P.$ – $J \to 0$ outside region ~ d^3 around $x'=0$)

$$= -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{x}'\left(\vec{\nabla}'\cdot\vec{J}(x')\right)$$

|| – Explain $\nabla\cdot J + \dot{\rho} = 0$ ↳ PTO

$$= -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\vec{x}'\left(i\omega\rho(x')\right)$$

$$= -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\,i\omega\,\vec{d}$$

N.B – notes use lower case $\vec{p}$

$\vec{D}$ – I don't want to confuse with!

where we have used charge conservation:-

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \; ; \qquad \vec{J}(\vec{x},t) = e^{-i\omega t}\,\vec{J}(\vec{x})$$

$$\dot{\rho}(\vec{x},t) = e^{-i\omega t}\,\rho(\vec{x})\,(-i\omega)$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{J}(\vec{x}) = i\omega\,\rho(\vec{x})$$

and definition of electric dipole moment $\vec{d} = \int d^3x'\,\vec{x}'\,\rho(\vec{x}')$

The time-dependent dipole moment

$$\vec{d}(t) = \int d^3x'\,\vec{x}'\,\rho(\vec{x}',t) = e^{-i\omega t}\,\vec{d}$$

$$\Rightarrow$$

$$\vec{A}_{far}(\vec{x},t) \simeq -\frac{\mu_0\, e^{ikr}}{r}\, i\omega\, e^{-i\omega t}\,\vec{d}$$

$$\boxed{\vec{A}_{far}(\vec{x},t) \simeq -\frac{\mu_0\, e^{ikr}}{r}\, i\omega\, \vec{d}(t).}$$

[physical field Re part.] *

$\therefore$ region far from source $\vec{J}$, leading order radiation produced just depends on electric dipole moment of source.

Clearly $\vec{A} \sim \frac{1}{r}$ for $r \gg \lambda \gg d$.

We can analyse the scalar field $\phi(\vec{x},t)$ in similar approximations:

$$\Box\Phi = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\frac{1}{\epsilon_0}\underbrace{\rho(\vec{x},t)}_{=\rho(\vec{x})e^{-i\omega t}}$$

retarded solution we found was:-

$$\Phi = \frac{1}{4\pi\epsilon_0}\int d^3x' \int dt' \frac{\rho(\vec{x}')e^{-i\omega t'}}{|\vec{x}'-\vec{x}|}\,\delta\Big(t' - \underbrace{\frac{|\vec{x}'-\vec{x}|}{c}}_{r/c + O(|\vec{x}'|/r)} - t\Big)$$

Far zone we can expand in powers of $|\vec{x}'|/r$, kr

(again assumed that $\rho(\vec{x}')$ localized inside a region $\sim d^3$)

So with $r = |\vec{x}|$; $t' \simeq t + r/c + O(|\vec{x}'|/r)$ via δ-fctn.

$$\Phi(r,t) \approx \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}')\,e^{-i\omega t'}}{r}$$

$$\approx \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}')\,e^{-i\omega t}}{r}$$

$$\Phi(r,t) = \Phi(r)\,e^{-i\omega t}$$

$$\Phi(r) \sim \frac{1}{4\pi\epsilon_0 r}\,Q$$

$Q = \int d^3x'\,\rho(\vec{x}')$ = total charge enclosed in region of size d^3.

Thus to leading order, this is just Coulomb field due to total (constant) charge of source.

– so $\boxed{\text{no radiation.}}$ But next order in expansion $|\vec{x}'|/r$ will yield dipole (radiation) term.

We can easily compute the corresponding $\vec{E}, \vec{B}$ fields from $\vec{A}, \Phi$:-

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad ; \quad \vec{E} = -\vec{\nabla}\Phi$$

or just use $\dot{\vec{E}} = c^2 \vec{\nabla} \times \vec{B}$ since in far region, $\vec{J} \simeq 0$.

Electric Dipole Radiation : $\vec{E}, \vec{B}$ fields

In far zone – expansion in $|\vec{x}'|/r$ gives rise to a 'multi-pole' series expansion (dipole is simplest -- higher poles involve $(|\vec{x}'|/r)^n$).

Let's calculate corresponding $\vec{E}, \vec{B}$ fields due to electric dipole of source.

use $\mu_0 i\omega = ick\mu_0 = \frac{1}{\epsilon_0}\epsilon_0\mu_0 ick$
$= \frac{1}{\epsilon_0} ik/c$

$$\vec{A}(x,t) = -\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left(\frac{ik}{c}\right)\vec{d}$$ (dipole radiation)

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{1}{4\pi\epsilon_0}\frac{ik}{c}\Big[\underbrace{\vec{\nabla}\Big(\frac{e^{ikr}}{r}\Big)}_{\vec{n}\frac{\partial}{\partial r}\ +\ \text{terms involving } \partial/\partial\theta,\ \partial/\partial\varphi}\Big]\times\vec{d}$$

$$\boxed{\vec{B}_{dipole} = +\frac{k^2}{4\pi\epsilon_0 c}\frac{e^{ikr}}{r}\left[1-\frac{1}{ikr}\right]\vec{n}\times\vec{d}(t)}$$ — Exact for dipole only //

what about $\vec{E}$?

well use $\dot{\vec{E}}(x,t) = \frac{\partial}{\partial t}\left(\vec{E}(\vec{x})e^{-i\omega t}\right) = c^2\vec{\nabla}\times\vec{B}(x,t)$
($\vec{J}=0$ in far zone)

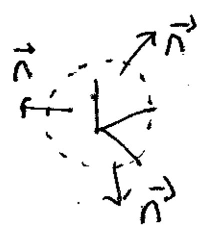
$$\therefore \quad \vec{E}(x,t) = \frac{ic^2}{\omega}(\vec{\nabla}\times\vec{B}) = \frac{ic}{k}\vec{\nabla}\times\vec{B}$$

$\Rightarrow$

$$\vec{E} = \frac{ic}{k}\vec{\nabla}\times\left[\frac{k^2 e^{ikr}}{4\pi\epsilon_0 r}\left(1-\frac{1}{ikr}\right)\vec{n}\times\vec{d}(t)\right]$$

To take curl, use identity $\nabla\times(f(r)\vec{C}) = (\vec{\nabla}f)\times\vec{C} + f\vec{\nabla}\times\vec{C}$ | $\vec{C} = \vec{n}\times\vec{d}$

N.B. although $|\vec{n}|=1$ it is not a constant vector



– direction changes

result (details in printed lecture notes)

$$\boxed{\vec{E} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}(\vec{n}\times\vec{d})\times\vec{n} + \frac{e^{ikr}}{4\pi\epsilon_0 r}\left(\frac{1}{r^2}-\frac{ik}{r}\right)\left(3\vec{n}(\vec{n}\cdot\vec{d}) - \vec{d}\right)}$$

Exact for dipole source.

- bit complicated but this and previous expression for $\vec{B}$ are exact for pure electric dipole case.

near field limit

$kr \ll 1$; $e^{ikr} \approx 1$, $[1-\frac{1}{ikr}] \approx \frac{-1}{ikr}$

$$\vec{B} \approx \frac{ik}{4\pi\epsilon_0 c}\frac{\vec{n}\times\vec{d}}{r^2}$$

$$\vec{E} \approx \frac{1}{4\pi\epsilon_0}\left[3\vec{n}(\vec{n}\cdot\vec{d}) - \vec{d}\right]\frac{1}{r^3}$$

- no radiation

(recall $\frac{1}{r^2}-\frac{ik}{r} = \frac{1}{r^2}(1-\underbrace{ikr}_{\ll 1})$ and $\frac{k^2}{r} \equiv \frac{k^2 r^2}{r^3} \ll \frac{1}{r^3}$)

Far zone $kr \gg 1$

$$\vec{B} \approx \frac{1}{4\pi\epsilon_0}\frac{k^2}{c}\frac{e^{ikr}}{r}\vec{n}\times\vec{d}$$

$$\vec{E} \approx \frac{k^2}{4\pi\epsilon_0}\,\frac{e^{ikr}}{r}\,(\vec{n}\times\vec{p}\times\vec{n}$$

$$= c\,\vec{B}\times\vec{n}\,.$$

We recognize the $\vec{E}$ and $\vec{B}$ fields in this zone as behaving like E.M. waves since

a) $\vec{E}\perp\vec{B}$ b) they are both $\perp$r to $\vec{n}$.

- So they describe outgoing spherical waves from the source. Also both fields fall off like $1/r$ (contrast with near field case).

Energy radiated by electric dipole.

We know energy transport (flux) is related to Poynting vector $\vec{P} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$.

Taking care of fact that physical $\vec{E},\vec{B}$ fields are $Re(\vec{E})$, $Re(\vec{B})$ in above expressions we have:-

$$\vec{S} = \frac{1}{\mu_0}\,Re(\vec{E})\times Re(\vec{B}).$$

$\vec{S}$ will depend on $\vec{x}, t$ in general.

Makes sense $\therefore$ to eg average over a single oscillation:-

trick

$$\left.\begin{array}{l} Re(\vec{E}) \\ \sim Re(e^{ikr}e^{-i\omega t}) \\ \sim \cos(kr-\omega t) \end{array}\right\}$$

$$\langle \vec{S} \rangle \equiv \frac{1}{T}\int_0^T \vec{S}(t)\,dt.$$

$$Re\,\vec{E}(\vec{x},t) = \cos\omega t\,\vec{E}(x,t=0) \;; \quad Re\,\vec{B}(x,t) = \cos\omega t\,\vec{B}(x,t=0)$$

$$\therefore \quad \langle \vec{S} \rangle = \frac{1}{\mu_0}\underbrace{\frac{1}{T}\int_0^T \cos^2\omega t\,dt}_{=\frac{1}{2}}\left(\vec{E}\times\vec{B}^*_{\vphantom{T}}\right)$$

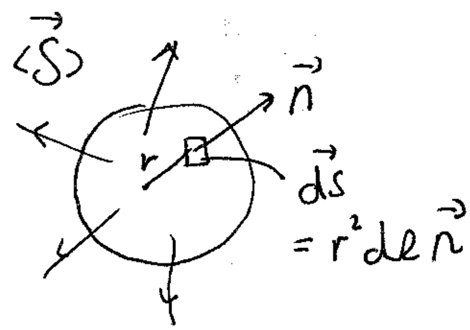
skip proof.

Sim: $Re\,\vec{B}$ $\sim \cos(kr-\omega t)$ $\int dt \to$ periodic ... drops out

$\Rightarrow e^{-i\omega t}\,e^{i\omega t}$

$$\boxed{\langle \vec{S} \rangle = \frac{1}{2\mu_0 c}|\vec{E}|^2\vec{n}}$$

this is real - so is correct phys. expression

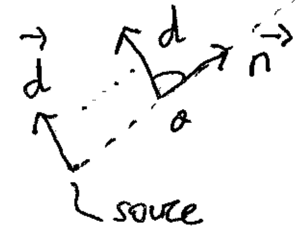


Thus power radiated through unit solid angle

$$\frac{dP}{d\Omega} \equiv r^2\vec{n}\cdot\langle\vec{S}\rangle = \frac{1}{2\mu_0 c}|r\vec{E}|^2$$

$$= \frac{1}{2\mu_0 c}\left|\left(\frac{k^2}{4\pi\epsilon_0}\vec{n}\times\vec{d}\right)\right|^2$$

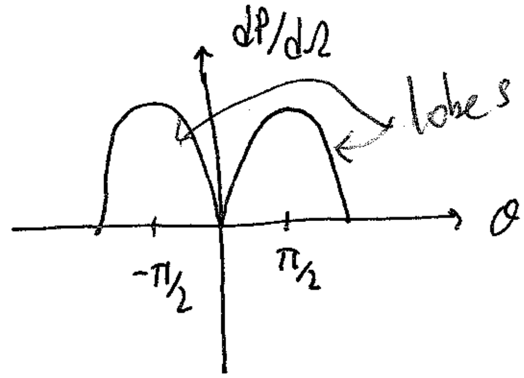
$$\frac{dP}{d\Omega} = \frac{1}{2}\frac{\mu_0}{16\pi^2 c}|\vec{d}|^2\omega^4\sin^2\theta$$

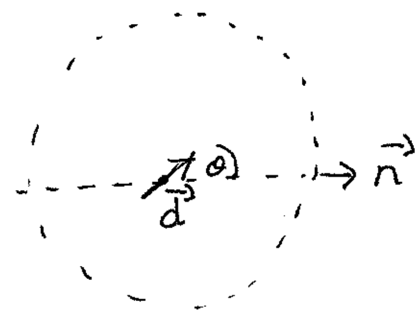


$(k^2c^2 = \omega^2$ ...$)$

the factor of $\sin^2\theta$ is typical angular

dependence on dipole radiation





** (notes wrong!)

- clearly power radiated is max at $\theta = \pi/2$ (or $-\pi/2$)

which corresponds to case where $\vec{n}$ is $\perp$r to

$\vec{d}$.

We can easily compute total power emitted by the

dipole:-

$$P = \int_{\text{sphere}} \frac{dP}{d\Omega}\, d\Omega = \frac{\mu_0}{32\pi^2 c}\, \omega^4 |\vec{d}|^2 \underbrace{\int_0^{\pi}\!\!\int_0^{2\pi} \sin^3\theta\, d\theta\, d\phi}_{8\pi/3}$$

$$\boxed{P = \frac{\mu_0}{12\pi c} |\vec{d}|^2 \omega^4}$$

ω^4 dependence ($\sim 1/\lambda^4$) also characteristic of electric dipole radiation.

Magnetic dipole radiation

So far we have studied radiation produced by ~~a~~ sources with electric dipole moments $\vec{d}$.

By including higher order terms in the expansions in $|\vec{x}'|/r$ (e.g. in far zone approximation) we shall see that magnetic dipole moments can also source radiation.

Also sometimes $\vec{d} = 0$ – so then these 'higher order' multipole terms may be the most important.

Going back to our general solution:-

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \vec{J}(\vec{x}') \, e^{ik|\vec{x}-\vec{x}'|} \frac{1}{|\vec{x}-\vec{x}'|}$$

recall:-

$$|\vec{x}-\vec{x}'| = r - \vec{n}\cdot\vec{x}' + O\left(\left(|\vec{x}'|/r\right)^2\right) \qquad (\vec{x} = r\vec{n})$$

In far zone, $kr \gg 1$

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = \frac{e^{ikr}}{r}\Big(\underbrace{1}_{\text{dipole contribution}} \; \underbrace{- ik\vec{n}\cdot\vec{x}'}_{\text{from } e^{ik|\vec{x}-\vec{x}'|}} + \underbrace{\tfrac{1}{2}\, n\cdot x'/r}_{\text{from } |\vec{x}-\vec{x}'|^{-1}} + \cdots\Big)$$

from $|\vec{x}-\vec{x}'|^{-1}$ – subleading to $k\vec{n}\cdot\vec{x}'$

The contribution beyond dipole approximation

is :-

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-ik) \int d^3x' \, \vec{J}(x') (\vec{n}\cdot\vec{x}')$$

Now use vector identity / trick

$$\vec{J}(x')(\vec{n}\cdot\vec{x}') = \frac{1}{2}\left[J(x')\,\vec{n}\cdot\vec{x}' + \vec{n}\cdot\vec{J}(x')\,\vec{x}' \right]^{*}$$
$$+ \frac{1}{2} \underbrace{\left[\vec{x}' \times \vec{J}(x') \right] \times \vec{n}}$$

Expanding - 2 terms - 1 cancels against *

-other gives $\frac{1}{2}(\vec{n}\cdot\vec{x}')\vec{J}(x')$

- this allows us to isolate separate contributions :-

look at last term : -

$$\vec{A}_{m.d}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-ik) \underbrace{\frac{1}{2} \int d^3x' \left(\vec{x}' \times \vec{J}(\vec{x}') \right)} \times \vec{n}$$

mag. dipole moment contribution.

$\vec{m} \equiv$ magnetic Dipole moment - see Wk 2.

$$\vec{A}_{m.d} = \frac{ik\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{n} \times \vec{m}$$

Let's calculate $\vec{E}, \vec{B}$ fields due to

$\vec{A}_{m.d}$: $\vec{B}_{m.d} = \vec{\nabla} \times \vec{A}_{m.d}$; $\vec{E}_{m.d} = \frac{ic}{k} \vec{\nabla} \times \vec{B}_{m.d}$

(in region outside source)

- we don't have to re-calculate from scratch because we can notice that

$$\vec{A}_{m.d}(\vec{m}) = \frac{i}{kc} \vec{B}^{e.d}(\vec{d} \to \vec{m})$$

- i.e. up to factors, $\vec{A}_{m.d}$ is just $\vec{B}$ due to electric dipole moment calculated earlier, with replacement $\vec{d} \to \vec{m}$.

so $\vec{B}_{m.d} = \vec{\nabla} \times \vec{A}_{m.d} = \frac{i}{kc} \vec{\nabla} \times \vec{B}^{e.d}(\vec{d} \to \vec{m})$

But $\vec{E}^{e.d}(\vec{d}) = \frac{ic}{k} \vec{\nabla} \times \vec{B}^{e.d}(\vec{d})$

$$\vec{B}_{m.d} = \frac{i}{\cancel{k}c} \frac{\cancel{k}}{ic} \vec{E}^{e.d}(\vec{d} \to \vec{m})$$

$$= \frac{1}{c^2} \vec{E}^{e.d}(\vec{d} \to \vec{m})$$

which is a neat result!

Similarly for $\vec{E}^{m.d}$:-

$$= \frac{ic}{k} \vec{\nabla} \times \vec{B}^{m.d} = \frac{ic}{k} \frac{1}{c^2} \vec{\nabla} \times \vec{E}^{e.d}(\vec{d} \to \vec{m})$$

$$= -\frac{1}{k^2} \vec{\nabla} \times (\vec{\nabla} \times \vec{B}^{e.d}(\vec{d} \to \vec{m}))$$

$$= -\frac{1}{k^2} \left(-\nabla^2 \vec{B}^{e.d}(\vec{d} \to \vec{m}) + \vec{\nabla} (\underbrace{\vec{\nabla} \cdot \vec{B}^{e.d}(\vec{d} \to \vec{m})}_{0}) \right)$$

$$= +\frac{1}{k^2} \left(\vec{\nabla} \times (\nabla^2 \vec{A}^{e.d}(\vec{d} \to \vec{m})) \right)$$

_ leave as exercise

But $\nabla^2 \vec{A}^{e.d}_{(x,t)} = \frac{1}{c^2} \frac{\partial^2 \vec{A}^{e.d}}{\partial t^2}(\vec{x})$

$$\Rightarrow \nabla^2 \vec{A}(\vec{x}) = -\frac{\omega^2}{c^2} \vec{A}^{e.d}(\vec{x})$$

$$\vec{E}^{m.d}(\vec{x}) = -\frac{1}{k^2} \frac{\omega^2}{c^2} \vec{\nabla} \times \vec{A}^{e.d}(\vec{d} \to \vec{m})$$

$$\boxed{\vec{E}^{m.d}(\vec{x}) = -\vec{B}^{e.d}(\vec{d} \to \vec{m})}$$ — State

Another neat result!

To summarize :-

$$\boxed{\begin{aligned} \vec{B}^{m.d} &= \frac{1}{c^2} \vec{E}^{e.d}(\vec{d} \to \vec{m}) \\ \vec{E}^{m.d} &= -\vec{B}^{e.d}(\vec{d} \to \vec{m}) \end{aligned}}$$

basically manifestation of the 'duality' symmetry behind Maxwell's Equations discussed in HW 1.

Electric Quadrupole Radiation

This contribution comes from the

terms (appearing at same order in kr expansion

as term that gave m.d. radiation)

in $\vec{J}(x')(\vec{n}\cdot\vec{x}') = \frac{1}{2}\left[\vec{J}(x')(\vec{n}\cdot\vec{x}') + \vec{n}\cdot\vec{J}(x')\vec{x}'\right] + \underbrace{\frac{1}{2}\left[\vec{x}'\times\vec{J}(x')\right]\times\vec{n}}_{\rightarrow \text{ m-d. contribution}}$

$$\vec{A}^{e.q.} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\frac{1}{2}\int d^3x'\left[\vec{J}(x')(\vec{n}\cdot\vec{x}') + \vec{n}\cdot\vec{J}(x')\vec{x}'\right]$$

use $\vec{\nabla}'\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$; $\vec{J}(\vec{x}',t) = \vec{J}(\vec{x}')e^{-i\omega t}$; $\rho(\vec{x},t) = \rho(\vec{x}')e^{-i\omega t}$

$$\Rightarrow \vec{\nabla}'\cdot\vec{J}(\vec{x}',t) = i\omega\rho(\vec{x}',t)$$

Skip details

$$\vec{A}^{e.q} \overset{I.P}{=} \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(+ik)\frac{1}{2}\int d^3x'\left(\vec{\nabla}'\cdot\vec{J}(x',t)\right)\vec{x}'(\vec{n}\cdot\vec{x}')$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(ik)(+i\omega)\underbrace{\frac{1}{2}\int d^3x'\,\vec{x}'(\vec{n}\cdot\vec{x}')\rho(x',t)}_{\text{electric quadrupole moment}}$$

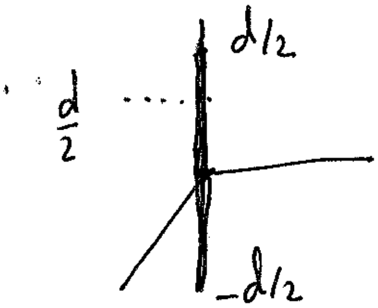
- 2 factors of $\vec{x}'$

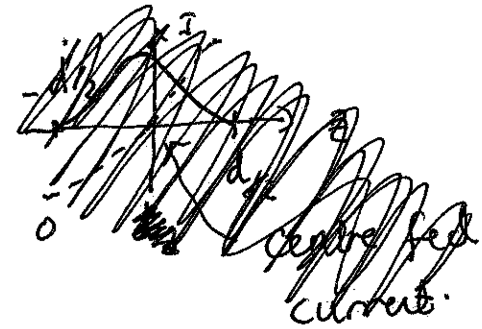
$$\boxed{\vec{A}^{e.q} = -\frac{ck^2\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{1}{2}\int d^3x'\,\vec{x}'(\vec{n}\cdot\vec{x}')}$$

Modelling Radiation Sources: Centre fed linear Antenna

Consider a model of $\vec{J}(\vec{x},t)$ provided by current carrying thin linear antenna oriented along z-axis

We can take $\underbrace{\vec{J}(\vec{x},t)}_{\text{current density}} = I \sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\,\delta(y)\,e^{-i\omega t}$





$$\vec{A}^{ant}(x,t) = \frac{\mu_0}{4\pi}\int d^3x'\, \vec{J}(x',t)\, \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

The time dependence of $\vec{A}(x,t)$ follows that of source $\sim e^{-i\omega t}$.

$$\vec{A}^{ant}(x) = \frac{\mu_0}{4\pi}\int d^3x'\, \vec{J}(x')\, \frac{e^{ikr}}{r}\, e^{-ik\vec{n}\cdot\vec{x}'}$$

where we have approximated $\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \approx \frac{e^{ikr}}{r}\, e^{-ik\vec{n}\cdot\vec{x}'}$

in far zone $kr \gg 1$

Define $\vec{x}'\cdot\vec{n} = z'\cos\theta$

(source $\vec{J}(x')$ only points along z-axis

so $\vec{x}' = (0,0,z')$) .

$$\vec{A}^{\text{out}} = \hat{z}\,\frac{\mu_0}{4\pi}\, I\, \frac{e^{ikr}}{r}\int_{-d/2}^{d/2} dz'\, \sin\left(\frac{kd}{2} - k|z'|\right) e^{-ikz'\cos\theta}$$

split $\int_{-d/2}^{d/2} \Rightarrow \int_0^{d/2} + \int_{-d/2}^{d}$

$$= \hat{z}\,\frac{\mu_0}{4\pi}\, I\, \frac{e^{ikr}}{r}\,\frac{2}{k}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

$\vec{B}^{\text{out}} \simeq ik\,\vec{n}\times\vec{A}^{\text{out}}$ to leading order in $1/r$

$\left(\vec{B}^{\text{out}} = \vec{\nabla}\times\vec{A}^{\text{out}} \;;\; \vec{\nabla} = \vec{n}\frac{\partial}{\partial r} + \cdots \quad \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{kr}\right) \simeq ik\frac{e^{ikr}}{kr} + O\left(\frac{1}{kr^2}e^{ikr}\right)\right)$

$|\vec{B}^{\text{out}}| = k\sin\theta\,|\vec{A}^{\text{out}}| = |\vec{E}|/c$ (Exercise to show this)

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = r^2\, n\cdot\langle\vec{S}\rangle = \frac{1}{2\mu_0 c}\,|r\vec{E}|^2$$

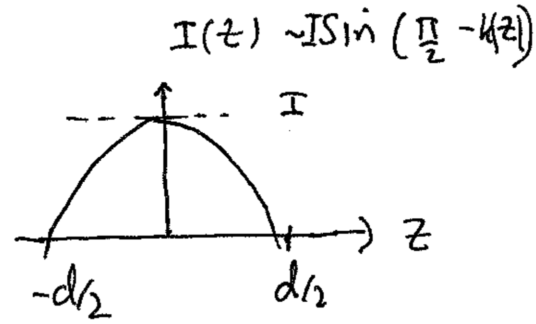
time averaged power/unit solid angle

$$= \frac{cr^2}{2\mu_0}\,\sin^2\theta\,|\vec{A}^{\text{out}}|^2$$

$$= \frac{\mu_0 c}{8\pi^2}\, I^2 \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\frac{kd}{2}}{\sin\theta}\right]^2$$

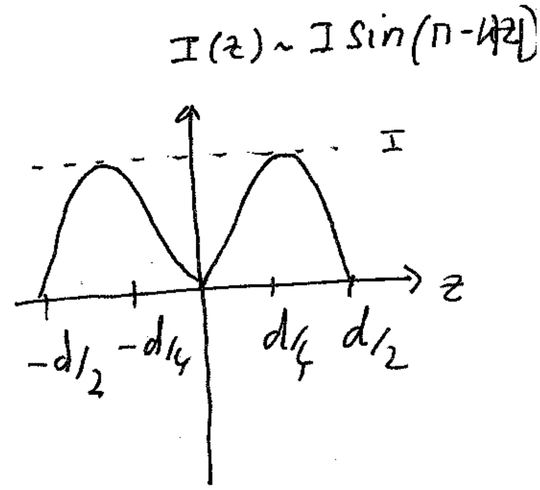
Special cases:

half-wave antenna: $kd = \pi$



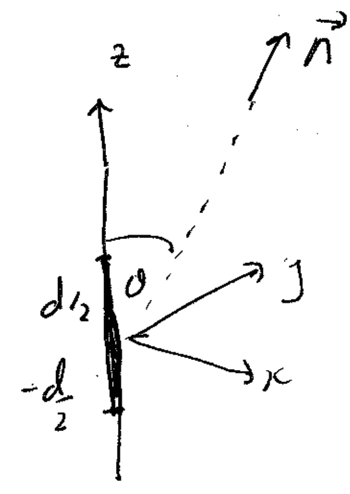
$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}.$$

full-wave antenna : $kd = 2\pi$



$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \frac{4\cos^4\left(\pi/2 \cos\theta\right)}{\sin^2\theta}$$

- angular distribution has changed



- By having more than one antenna - (ie many sources)
- can get complex patterns of $\frac{dP}{d\Omega}$