Solving Maxwell's Equations : Green Functions

Maxwell Eqn's written in terms of $\vec{E}, \vec{B}$ mixed set of coupled 1st order partial diff^l equations. To apply method of Green's functions - better if we can rewrite these equations as 'unmixed' set of linear diff^l equations.

To do this we introduce concept of Scalar and Vector potentials

In vacuo, with sources

$$① \quad \vec{\nabla}\cdot\vec{B} = 0 \qquad ③ \quad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$② \quad \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad ④ \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

① Can be solved by writing $\vec{B} = \vec{\nabla}\times\vec{A}$ 'vector' potential

④ " " " $\vec{E} = -\vec{\nabla}\Phi + \frac{\partial \vec{A}}{\partial t}$

which is nice but note there is a 'redundancy' in these solutions: namely

$$\vec{A}(x,t) \to \vec{A}(x,t) + \vec{\nabla}\Lambda(x,t)$$
$$\Phi(x,t) \to \Phi(x,t) - \frac{\partial \Lambda(x,t)}{\partial t}$$

leaves $\vec{E}$ and $\vec{B}$ unchanged!

Infact this is an example of a <u>gauge symmetry</u>

- and Maxwell's theory is indeed a gauge theory.

that can be written in a manifestly Lorentz invariant way

through the 4-vector potential $A_\mu = (\vec{A}, -\frac{\Phi}{c})$ - more later.

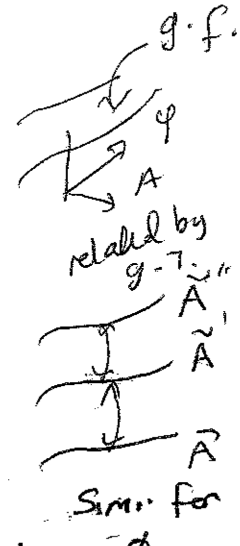
For now - let's consider remaining 2 Maxwell Equations:

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\Phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \nabla^2\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

$$= \frac{1}{c^2}\left(-\vec{\nabla}\frac{\partial\Phi}{\partial t} - \frac{\partial^2\vec{A}}{\partial t^2}\right) + \mu_0\vec{J}$$

$$\Rightarrow \quad \nabla^2\vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J}$$



One way of 'fixing' the redundancy previously mentioned is to impose a gauge fixing condition - i.e. a constraint on $\vec{A}, \Phi$ which is not itself invariant under the gauge transformation.

There is no unique way to ~~fi~~ choose gauge fixing conditions - many different gauge choices - but we expect all give rise to same physical $\vec{E}$ and $\vec{B}$ fields in any given physical problem.

Example: Lorentz gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$

$\partial_\mu A^\mu = 0$

The Maxwell eqn's simplify :-

$$\Box\Phi \equiv \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\epsilon_0$$

$$\Box\vec{A} \equiv \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

– LHS involves the 'klein-Gordon' operator $\nabla^2 - \frac{1}{c^2}\partial^2/\partial t^2$ for both A^μ, Φ which is suggestive (see later) that these can be combined into single 4-vector $(\vec{A}, \Phi)$ – See later on in course.

Another possible gauge fixing choice is 'Coulomb gauge'

$\vec{\nabla}\cdot\vec{A} = 0$ – not lorentz invariant – (not a problem)

Again Maxwell eq's simplify :-

$$\boxed{\begin{aligned}\nabla^2\Phi &= -\rho/\epsilon_0\\ \nabla^2\vec{A} &= -\mu_0\vec{J_t}\end{aligned}}$$

$\vec{J_t}$ is transverse electric current satisfying $\vec{\nabla}\cdot\vec{J_t} = 0$

$$J_t = \left(\vec{J} - \epsilon_0\vec{\nabla}\frac{\partial\Phi}{\partial t}\right) \quad ; \quad \vec{\nabla}\cdot\vec{J_t} = \vec{\nabla}\cdot\vec{J} - \epsilon_0\nabla^2\frac{\partial\Phi}{\partial t}$$

$$= \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}$$

(as follows from Maxwell eqn)

$= 0$ by charge conservation)

Before discussing Green's function methods -

need to remind ourselves some properties of Dirac Delta function.

- Can think of 1 d delta fct^n as continuum limit of kronecker delta
$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}.$$

$$\sum_m \delta_{nm} f_m = f_n.$$

$$\delta(x-x') = \begin{cases} 0 & x \neq x' \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-x') f(x) dx = f(x') \quad ; \quad \delta(x-x') = \delta(x'-x)$$

$$\therefore \int_{-\infty}^{\infty} \delta(x-x')\, dx = 1 \quad \text{or indeed} \quad \int_a^b \delta(x-x') dx = 1 \quad \text{if } x = x' \in [b,a].$$

propeties: $\delta(cx) = \frac{1}{|c|}\delta(x)$. (just change variables)

$$\delta(x^2-a^2) = \frac{1}{2|a|}\left[\delta(x-x) + \delta(x+a)\right]$$

(See printed notes for proof.)

Important representation of δ-fct^n is as a F.T. of identity :-
$$\delta(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, 1.\, e^{-i\lambda\omega}$$

Let's check: $\int_{-\infty}^{\infty} d\lambda\, \overset{\text{"}\delta(\lambda'-\lambda)\text{"}}{\delta(\lambda-\lambda')}\, f(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\omega\, f(\lambda)\, e^{+i(\lambda-\lambda')\omega}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\lambda'\omega}\, f(\omega) \qquad \left(f(\omega) = \int d\lambda\, e^{i\lambda\omega} \right)$$

$$= f(\lambda') \quad \checkmark$$

In more than 1-d e.g 3.

$$\delta^{(3)}(\vec{x}-\vec{x}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z') \quad \text{— Cartesian coords.}$$

In e.g spherical polar coords:

$$\delta^{(3)}(\vec{r}-\vec{r}') = \frac{1}{r^2 \sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$

$$\left(\int_0^\infty r^2 dr \int_0^{4\pi} d\Omega\, \delta^{(3)}(\vec{r}-\vec{r}') = 1 \qquad d\Omega = \sin\theta\, d\theta\, d\phi \right)$$

Green's function:

$\mathcal{L}_x f(x) = h(x)$. — $h(x)$ is given

"source" — $\mathcal{L}_x$ = linear operator.

Solution $\quad f(x) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} G_{\mathcal{L}}(x-x')\, h(x')\, dx'$

where $G_{\mathcal{L}}(x-x')$ — Green fct^n associated to Lin op $\mathcal{L}_x$

$$\mathcal{L}_x\, G(x-x') = -4\pi\,\delta(x-x').$$

[So then it's easy to see how $f(x)$ is indeed the solution].

- Extends to functions in several variables.

E.g. $\nabla^2 \Phi(\vec{x}') = \rho(\vec{x}')/\epsilon_0$

- Solution: $\Phi(\vec{x}) = \frac{-1}{4\pi\epsilon_0} \int G(\vec{x}-\vec{x}')\, \rho(\vec{x}-\vec{x}')\, dx'$

where $\nabla^2 G(\vec{x}-\vec{x}') = \underbrace{-4\pi}_{\text{convention}} \delta^{(3)}(\vec{x}-\vec{x}')$. is general solution of Coulomb ~~field~~ potential.

So what is $G(\vec{x}-\vec{x}')$?

Well lets be a little more general. Working with Lorentz gauge:

$$\Box \vec{A} = -\mu_0 \vec{J}.$$

$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. This again may be solved using Green fctn's. From this well be able to find the Green function of operator ∇^2 by restriction.

First work with F.T. $\vec{A}(\vec{x},\omega)$

$$\vec{A}(x,\omega) \equiv \int dt\, \vec{A}(\vec{x},t)\, e^{i\omega t}$$

$$\Rightarrow \quad (\nabla^2 + k^2)\vec{A}(\vec{x},\omega) = -\mu_0 \vec{J}(\vec{x},\omega).$$

$k^2 = \omega^2/c^2$. We want to find $G_k(\vec{x},\vec{x}')$ that solves:

$$\boxed{(\nabla^2 + k^2)\, G_k(\vec{x},\vec{x}') = -4\pi\delta^{(3)}(\vec{x}-\vec{x}')}$$

effectively "identity" on function space

basically "inverse" of $\nabla^2 + k^2$

Then, $$A(\vec{x},\omega) = \frac{\mu_0}{4\pi}\int G_k(\vec{x},\vec{x}')\, J(x',\omega)\, d^3x'$$

- Easy to check. Hit LHS $\vec{\nabla}_x^2 + k^2$

$$(\nabla_x^2 + k^2)\vec{A}(\vec{x},\omega) = \frac{\mu_0}{4\pi}\int \underbrace{(\nabla_x^2 + k^2)\, G_k(\vec{x},\vec{x}')}_{= -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')}\, J(x',\omega)\, d^3x'$$

$$= -\mu_0 J(x',\omega) \quad \checkmark$$

Now since $\nabla^2 + k^2$ is translation invariant: $(\vec{x} \to \vec{x} + \vec{c})$

$\Rightarrow$ $G_k(\vec{x},\vec{x}')$ only depends on difference $(\vec{x}-\vec{x}')$

Furthermore it's rotation invariant (since ∇^2 is)

$\Rightarrow$ G_k can only depend on $|\vec{x}-\vec{x}'| \equiv r$

if we work in spherical polar co-ords:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \text{terms involving } \partial/\partial\theta,\ \partial/\partial\varphi$$

But $G_k(r,\theta,\varphi) = G_k(r)$

$$\therefore\ \nabla^2 G_k = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) G_k(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r G_k(r)\right)$$

$$\Rightarrow \quad \frac{1}{r}\frac{d^2}{dr^2}\left(r G_k(r)\right) + k^2 G_k(r) = -4\pi\,\delta^{(3)}(\vec{r}). \quad *$$

Take $r \neq 0$. rhs $\equiv 0$.

$$\frac{d^2}{dr^2}\left(r G_k(r)\right) + k^2 (r G_k) = 0.$$

$$\Rightarrow \quad r G_k = A e^{ikr} + B e^{-ikr}, \quad A, B \text{ constants}.$$

As $r \to 0$, $\frac{1}{r}\frac{d^2}{dr^2}\left(r G_k\right)$ term dominates LHS $*$

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r G_k(r)\right) = -4\pi\,\delta^{(3)}(\vec{r})$$

– Looks like Poisson Eqn $\nabla^2 \Phi(r) = -\rho/\epsilon_0$

with $\rho(\vec{r}) = 4\pi\epsilon_0 \delta^{(3)}(\vec{r})$; $\Phi(\vec{r}) = G_k$

point charge $q = 4\pi\epsilon_0$
located at origin $\vec{r} = 0$.

But we know solution: $\Phi(r) = \frac{q}{4\pi\epsilon_0 r}$

for point charge q at origin.

$$\therefore \quad G_k = \frac{4\pi\epsilon_0}{4\pi\epsilon_0 r} = \frac{1}{r}. \qquad \text{as } r \to 0.$$

So we learn that $r G_k(r) = A e^{ikr} + B e^{-ikr}$ $\quad r \neq 0$

and as $r \to 0$; $G_k(r) \to 1/r$. $\Rightarrow$ $A + B = 1$.

$$\left[\begin{array}{c} G_k(r) = \frac{Ae^{ikr}}{r} + \frac{Be^{-ikr}}{r} \\ A + B = 1 \end{array} \right]$$

- In any given physical problem, expect boundary conditions away from $r=0$ to determine other condition on A, B.

For now let's take $G_k(r) = A\,G^{(+)}(r) + B\,G^{(-)}(r)$

$$\boxed{G^{(\pm)}(r) \equiv \frac{e^{\pm ikr}}{r}}$$

Going back to solution we seek for

$\vec{A}(\vec{x},t)$ recall by inverse f.t.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \overset{\overset{-\text{ sign}}{\downarrow}}{e^{-i\omega t}}\, A(\vec{x},\omega)$$

Since $\vec{A}(x,\omega)$ involve $G_k^{\pm}$ we will find

combination: $$e^{-i\omega t}\,\frac{e^{\pm ikr}}{r} = \frac{e^{-i(\omega t \mp kr)}}{r}$$

G_k^{+} is an outgoing wave (wrt $\vec{r}=0$)

→ usual physical choice. [outgoing from "source"]

G_k^{-} — represents incoming wave.

∴ Normal physical choice will be $B=0$.

$$\vec{A}(\vec{x},\omega) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}',\omega)\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x'$$

$$\Rightarrow \vec{A}(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\frac{\mu_0}{4\pi}\int d^3x'\,dt'\,\vec{J}(\vec{x}',t') \times \frac{e^{ik|\vec{x}-\vec{x}'|}}{|x-x'|}\,e^{i\omega t'}$$

[where we use $\vec{J}(\vec{x},\omega) \equiv \int_{-\infty}^{\infty} dt'\, e^{i\omega t'}\, J(\vec{x},t')$].

ω – integration gives: (recall $k = \omega/c$).

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')}\, \frac{e^{i\frac{\omega}{c} r}}{r} = \frac{1}{2\pi r}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(r/c-\tau)}$$

$$\tau \equiv t-t' \qquad\qquad = \frac{1}{r}\,\delta(\tau - r/c).$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int_{-\infty}^{\infty} dt'\, \vec{J}(x',t')\,\frac{1}{r}\,\delta(t'-(t-r/c))$$

$$\left(\delta(\tau - r/c) = \delta(r/c - \tau)\right)$$

$$\boxed{\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\, \vec{J}(x', t-r/c)\,\frac{1}{r}} \qquad r = |\vec{x}-\vec{x}'|$$

points to note:– since $r \geq 0$, RHS has source at times $t' \leq t$, for all points $\vec{x}'$ in $\int$.

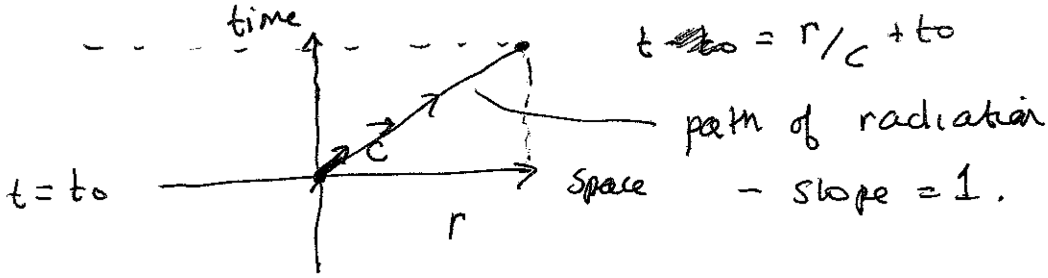
Interpretation: Take simple example of a localized current being switched on at $\vec{x}' = 0$ ~~[illegible]~~ a time

$t = t_0$ $\Big|\ \vec{J}(\vec{x},t) = \vec{C}\,\delta^{(3)}(\vec{x})\,\theta(t-t_0)$

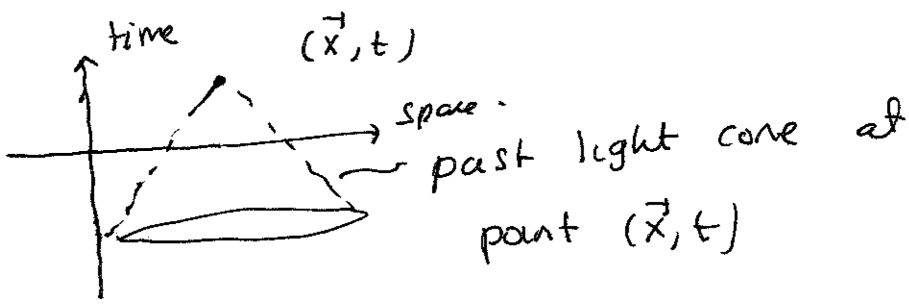
$$\Rightarrow \vec{J}(x', t-r/c) = \vec{C}\,\delta^{(3)}(\vec{x}')\,\theta(t - r/c - t_0)$$

$$\Rightarrow \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\,\frac{\vec{C}}{r}\,\theta(t-r/c-t_0)$$

$$= \begin{cases} 0 & t-t_0 \leq r/c \\ 1 & t-t_0 \geq r/c \end{cases}$$



- So general solution encodes 'causality' and the fact that upon switching the current on at time $t = t_0$, it takes a time $r/c + t_0$ before 'disturbance' (i.e. radiation) reaches point $|\vec{x}| = r$.



- In general for sources which are not necessarily localized in space - they will contribute to $\vec{A}(\vec{x}, t)$ if their history is such that they lie on the past light cone of point $(\vec{x}, t)$.

- Potentials $\vec{A}(\vec{x}, t)$ 'solved' in this way are called 'retarded' potentials - and the Green function $G_k^+(r, \omega)$ - retarded Green functions.

Solutions constructed using $G_k^-(r, \omega)$ function are called 'advanced' and involve $\vec{J}(\vec{x}', t + r/c)$

- represent incoming waves

We can solve for $\Phi(\vec{x},t)$ in similar way

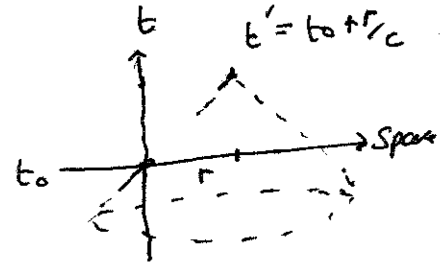
for retarded field:

$$\boxed{\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} d^3x' \, \rho(\vec{x}', t - r/c) \frac{1}{r}}$$

For a point like charge q placed at say $\vec{x}' = 0$

at time $t = t_0$;
$$\rho(\vec{x}', t) = q \, \delta^{(3)}(\vec{x}') \, \theta(t - t_0)$$

$$\Rightarrow \quad \Phi(\vec{x},t) = \frac{q}{4\pi\varepsilon_0 r} \, \theta(t - r/c - t_0)$$



- so again see causality encoded by Green function!