Electromagnetic Wave; Polarized Waves; dispersion

Plane Wave Solutions:

$$\vec{\nabla}\cdot\vec{D} = 0 \quad ; \quad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{H} = +\frac{\partial \vec{D}}{\partial t} \quad ; \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(consider source free case - later we will discuss radiation from sources)

Assuming isotropic, linear medium $\vec{D} = \epsilon\vec{E}$; $\vec{H} = \frac{1}{\mu}\vec{B}$

Following can be verified as solving above eqs (Ex-!)

$$\vec{E} = \vec{E}_0\, e^{i\vec{k}\cdot\vec{n}\cdot\vec{x} - i\omega t} \quad ; \quad \vec{B} = \vec{B}_0\, e^{ik\vec{n}\cdot\vec{x} - i\omega t}$$

if
$$k^2 = \mu\epsilon\omega^2 \qquad (1)$$
$$n\cdot\vec{E}_0 = \vec{n}\cdot\vec{B}_0 = 0 \qquad (2)$$
$$\vec{B}_0 = \vec{n}\times\vec{E}_0\sqrt{\mu\epsilon} \qquad (3)$$

with $\vec{E}_0$, $\vec{B}_0$ constant vectors (real).

$\vec{n}$ is taken to be unit vector (by scaling k)

= direction of wave propagation in space.

[[physical $\vec{E}$, $\vec{B}$ given by real parts of complex fields $\vec{E}$, $\vec{B}$ above]]

$k = 2\pi/\lambda$ $\quad \lambda =$ wave-length $\quad k$ is wave number

$\omega = 2\pi f = \frac{2\pi}{T}$ as usual.

Speed of wave $v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = c/n$ = phase velocity.

$n =$ refractive index $= \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$, $\quad c \equiv \frac{1}{\sqrt{\mu_0\epsilon_0}}$ = speed light in vacua.

$\vec{E}, \vec{B}$ are both $\perp$r to direction of propagation $\vec{n}$

and by virtue of (3) $\vec{E} \perp \vec{B}$.

Direction of energy flow ~~per~~ area is along Poynting's vector $\vec{P}$

$$\langle \vec{P} \rangle_{\text{wave}} = \frac{1}{\mu}\langle \vec{E}\times\vec{B} \rangle_{\text{wavelength}} = \sqrt{\epsilon/\mu}\,(\vec{E}_0\cdot\vec{E}_0)\,\vec{n}$$

Here $\langle\ \rangle_{\text{wavelength}}$ means averaging quantity along 1 cycle (= 1 wavelength)

$$Re\,\vec{E} = \vec{E}_0 \cos(k(\vec{n}\cdot\vec{x}) - \omega t) \;; \quad Re\,\vec{B} = \vec{B}_0 \cos(k(\vec{n}\cdot\vec{x}) - \omega t)$$

$$\frac{1}{\mu}\langle \vec{E}\times\vec{B} \rangle = \sqrt{\epsilon/\mu}\,(\vec{E}_0\cdot\vec{E}_0)\,\vec{n}\,\frac{1}{\lambda}\int_0^\lambda \cos^2(k(\vec{n}\cdot\vec{x}) - \omega t)\, d(\vec{n}\cdot\vec{x})$$

So $\vec{n}$ is direction of energy flow.

$$= \frac{1}{2}\sqrt{\epsilon/\mu}\; E_0^2\,\vec{n}.$$

(could also average over single period T $\quad \frac{1}{T}\int_0^T \vec{E}\times\vec{B}\, dt$ — same answer)

notes missing this.

Similarly we can calculate the time averaged (over 1 cycle) energy density

$$\mathcal{E} = \frac{1}{2}\epsilon_0 \int_0^T dt(\vec{E}\cdot\vec{E}) + \frac{1}{2\mu}\int_0^T dt(\vec{B}\cdot\vec{B})$$

$$= \left(\frac{\epsilon}{2}\vec{E}_0\cdot\vec{E}_0 + \frac{\epsilon}{2}\vec{E}_0\cdot\vec{E}_0\right)\int_0^T \cos^2(k(n.x)-\omega t)\frac{dt}{T}$$

$$= \frac{1}{2}\epsilon(\vec{E}_0\cdot\vec{E}_0).$$

[notes missing factor of $\frac{1}{2}$]

Polarized Plane Waves.

A more enlightening way of writing previous solution is to introduce 3 mutually orthogonal unit vectors $(\vec{e}_1, \vec{e}_2, \vec{n})$. $\vec{e}_1 \parallel \vec{E}_0$; $\vec{e}_2 \parallel \vec{B}_0$.

Above solution $\vec{E}_0 = E_0\vec{e}_1$; $\vec{B}_0 = B_0\vec{e}_2 = \sqrt{\mu\epsilon}E_0\vec{e}_2$.

$E_0 = \sqrt{\vec{E}_0\cdot\vec{E}_0}$.

It's easy to verify that $\vec{E}_0 = E_0'\vec{e}_2$; $B_0 = -E_0'\sqrt{\mu\epsilon}\,\vec{e}_1$ also a solution of Maxwell Eqs ($\vec{E}_0, \vec{B}_0$ still $\perp$r to $\vec{n}$ and to eachother). This is just 90° rotation of previous solution.

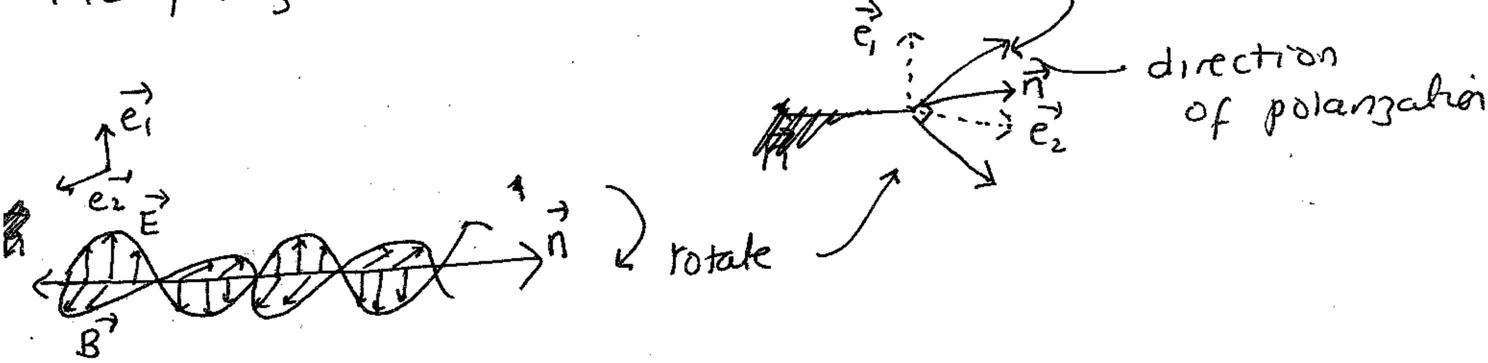
These solutions are called <u>linearly polarized</u> with the direction of polarization given by direction of $\vec{E}$. Most general linearly polarized plane wave solution is just linear combination of these :-

$$\vec{E} = \left(E_1 \vec{e}_1 + E_2 \vec{e}_2\right) e^{ik(\vec{n}\cdot\vec{x} - \omega t)}$$

$$\left(\vec{B} = \sqrt{\mu\epsilon}\,\left(E_2 \vec{e}_1 - E_1 \vec{e}_2\right) e^{ik(\vec{n}\cdot\vec{x} - \omega t)} \right)$$

with E_1, E_2 arb. real constants.

The polarization direction is given by $\left(E_1 \vec{e}_1 + E_2 \vec{e}_2\right)$



There is another kind of polarized wave solution which we can see if we allow E_1, E_2 above to be complex constants (there is no restriction on them to be real in order to solve Maxwell's Eqs)

E.g. $E_2 = i E_1$ $\qquad \vec{E} = E_1 \left(\vec{e}_1 + i \vec{e}_2\right) e^{ik(\vec{n}\cdot\vec{x} - \omega t)}$

$\vec{B} = \ldots$

Physical electric field $= \mathrm{Re}(\vec{E})$

5

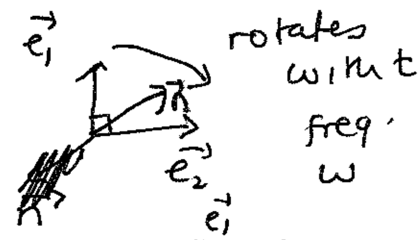
$$\vec{E}_{phys} = E_1 \Big(\cos(k\vec{n}\cdot\vec{x} - \omega t)\,\vec{e}_1 + \underbrace{\sin(k\vec{n}\cdot\vec{x} - \omega t)\,\vec{e}_2}_{\text{phase shift by } \pi/2} \Big)$$

*

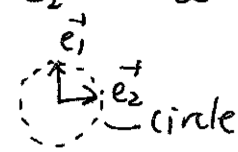
no phase shift $\Rightarrow$ linearly polarized
$\pi/2 \Rightarrow$ circular
arb $\Rightarrow$ elliptically.

$\vec{B}_{phys} = \ldots$

So what is direction of polarization? Well for given point ($\vec{n}\cdot\vec{x}$ fixed) direction of $\vec{E}$ is time-dependent $= \cos(\alpha - \omega t)\,\vec{e}_1 + \sin(\alpha - \omega t)\,\vec{e}_2$

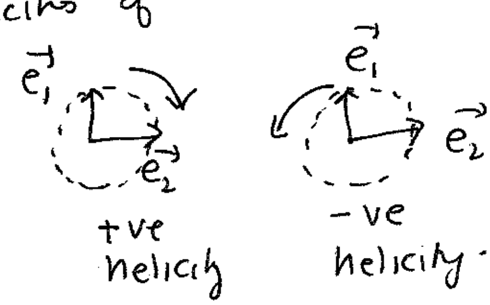


$\therefore$ direction of polarization rotates in time – with frequency ω.



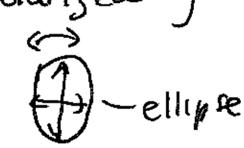
$\Rightarrow$ <u>circularly polarized wave</u>.

There are 2 distinct kinds $\leftrightarrow$ 2 helicities of clockwise / anti clockwise rotation



Most general solution is a linear combination of these.

$\Rightarrow$ Full set of polarized wave solutions:
4 real parameters (2 linearly polarized, 2 circularly polarized)



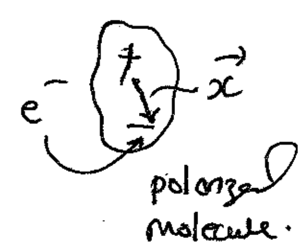
*

Dispersion

- This describes general class of phenomena whereby the medium behaves differently for waves of different frequency passing through it. E.g. parameters ϵ, μ have thus far been assumed to be independent of frequency even if they may vary depending on type of media.

We will now consider a very simple model in which dispersion can be understood and which results in $\epsilon = \epsilon(\omega)$.

Basically assume that material is made of molecules localised at e.g. lattice sites in case of solid. Let's just take a single electron in each molecule as being under a restoring force $\vec{F} = -m\omega_0^2 \vec{x}$, m = mass of electron, ω_0 the oscillation freq.



Equation of motion of electron, if we apply an electric field $\vec{E}(\vec{x},t)$ is :-

$$\frac{d^2\vec{X}}{dt^2} + \gamma \frac{d\vec{X}}{dt} + \omega_0^2 \vec{X} = -\frac{e}{m} \vec{E}(\vec{x},t)$$

Where we have included a damping term $\gamma \frac{d\vec{X}}{dt}$ in above equation

(i.e. model of e^- in polarized molecule is of a damped SHO)

Imagine a simple case where we apply constant $\vec{E}$ field to material. Equation can be solved if we look in first instance for oscillatory solutions:

$\vec{x}(t) = \vec{x}_0 e^{-i\omega t}$, the electron contributes a dipole moment to the molecule :-

$$\vec{p} = -e\vec{x}_0 = \frac{e^2}{m}(\omega_0^2 - \omega^2 - i\omega\gamma)^{-1} \vec{E}.$$

Recall $\vec{p} = \epsilon_0 \chi_e \vec{E}$ for small fields $\vec{E}$,

$$\therefore \quad \epsilon = \epsilon_0(1+\chi_e) = \epsilon_0 + \frac{e^2}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)}.$$

Note: physical dielectric constant $\epsilon_{phys} = Re(\epsilon)$ and is responsible for dispersion: Imaginary part is related to absorbtion of energy by the medium and so to 'dissipation'

$\gamma \ll \omega_0$ in physical examples; (ω_0 called resonant frequency of dielectric) so $Im(\epsilon)$ is small.

If $\omega < \omega_0$; $\epsilon(\omega)/\epsilon_0 > 1$

since (if $\omega\gamma \ll 1$) $\epsilon/\epsilon_0 \approx 1 + \frac{e^2}{m}\frac{1}{\underbrace{(\omega_0^2-\omega^2)}_{>0}} > 1$

As $\omega \to \omega_0$ from below, there is cancellation in denominator term in $\epsilon(\omega)$ and it becomes dominated by pure imaginary term:-

$$\epsilon/\epsilon_0 \overset{\omega \to \omega_0}{\approx} 1 + \frac{e^2}{m} \frac{i}{\omega_0 \gamma} \qquad (\gamma \ll \omega_0^{-1})$$

This is indicative of energy dissipation (absorption) by medium - 'resonant absorption' - see why later.

To understand physically better the model, need to introduce continuous Fourier transform for relation $f(\vec{x}, \omega) \Leftrightarrow f(\vec{x}, t)$

Consider any suitably well behaved function $F(\vec{x}, \omega)$. It's fourier transform is given by $F(\vec{x}, t)$ where

$$F(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\vec{x}, \omega)\, e^{-i\omega t}\, d\omega$$

integration over frequency space.

The functions $e^{i\omega t}$ form an orthogonal basis, that is:-

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}\,d\omega = \delta(t-t').$$

$\delta(t)$ is 1-d Dirac delta function. Really a 'distribution' rather than a standard function.

properties:

$$\int_{-\infty}^{\infty} \delta(t-t')\,f(t)\,dt = f(t')$$

for any function $f(t)$. (More next week)

Elementary to show that inverse fourier transform is:-

$$\tilde{F}(\vec{x},\omega) = \int_{-\infty}^{\infty} F(\vec{x},t)\,e^{i\omega t}\,dt$$

An important property of the F-T. we will need is convolution Theorem.

If we have 2 functions $f(\omega)$ and $g(\omega)$ then it's f-t, $h(t)$ is given by:-

$$\boxed{H(t) = \int_{-\infty}^{\infty} F(t')\,G(t-t')\,dt'}$$

$$F(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\,e^{-i\omega t}\,d\omega; \quad G(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)\,e^{-i\omega t}\,d\omega.$$

Kramers - Kronig relation:

Let's consider the physical displacement field

$\vec{D}(\vec{x},t)$ due to '1-electron' model of polarized molecule

For small fields (linear media)

$$\vec{D}(\vec{x},\omega) \equiv \underbrace{\epsilon_0 \epsilon_r(\omega)}_{\epsilon(\omega)} \vec{E}(\vec{x},\omega)$$

We have previously found that $\epsilon/\epsilon_0 = 1 + \dfrac{e^2/m\epsilon_0}{\omega_0^2-\omega^2-i\omega\gamma} \equiv \epsilon_r(\omega)$

- So ϵ_r is frequency dependent.

Using convolution thm of F.T.

$$D(\vec{x},t) = \epsilon_0 \vec{E}(x,t) + \epsilon_0 \int_{-\infty}^{\infty} G(\tau)\, \vec{E}(\vec{x},t-\tau)\, d\tau$$

where
$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\epsilon_r(\omega)-1]\, e^{-i\omega\tau}\, d\omega$$

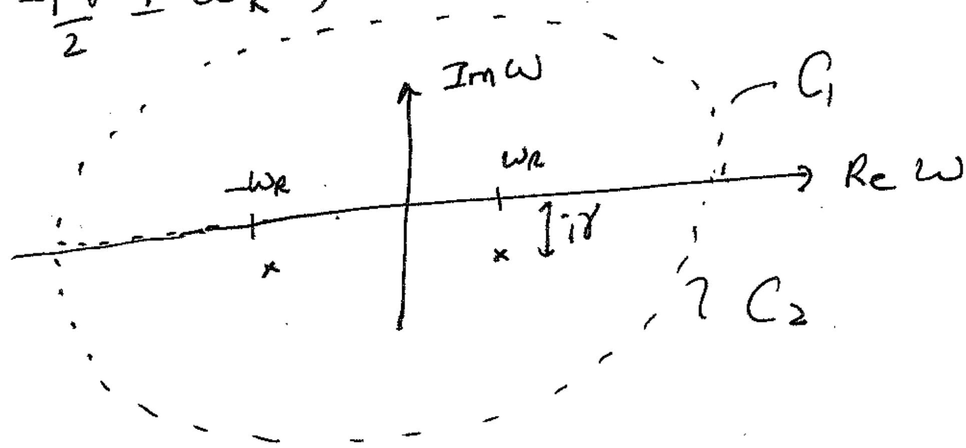
with
$$\epsilon_r(\omega) - 1 = \frac{\omega_p^2}{\omega_0^2-\omega^2-i\gamma\omega} \quad ; \quad \boxed{\omega_p^2 \equiv e^2/m\epsilon_0}$$

$$\rightarrow \quad G(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega_0^2-\omega^2-i\gamma\omega}\, d\omega$$

Use Cauchy's Thm to evaluate:

$$\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \equiv \frac{-1}{(\omega-\omega_+)(\omega-\omega_-)}$$

$$\omega_\pm = -\frac{i\gamma}{2} \pm \omega_R \; ; \qquad \omega_R^2 = \omega_0^2 - \gamma^2/4 .$$



C_1, C_2 both closed contours where Im $\omega \to +\infty$ or $-\infty$.

For Im $\omega > 0$, $e^{-i\omega\tau} \to 0$ as Im $\omega \to \infty$ if $\underline{\tau < 0}$.

Then, by Cauchy Thm, $\int_{-\infty}^{\infty} (\quad) d\omega = \int_{C_1} (\quad) d\omega$

$= 0$ since no poles in upper half plane.

$\therefore$ $\boxed{G(\tau) = 0 \qquad \tau < 0.}$

For $\tau > 0$, have to choose contour C_2 — which now encloses 2 poles at $\omega = \omega_\pm$.

The result is:-

$$G(\tau) = -\frac{\omega_p^2}{\cancel{2\pi}} \cancel{2\pi} i \left[\frac{e^{-i\omega_+ \tau}}{-(\omega_+ - \omega_-)} + \frac{e^{-\omega_- \tau}}{-(\omega_- - \omega_+)} \right]$$

$$= -\omega_p^2 i \, e^{-i(-i\gamma\tau/2)} \left[\frac{e^{-i\omega_R \tau}}{-2\omega_R} + \frac{e^{i\omega_R \tau}}{2\omega_R} \right]$$

$$\boxed{G(\tau) = \omega_p^2 \, e^{-\gamma\tau/2} \frac{\sin \omega_R \tau}{\omega_R}} \quad \tau \geq 0$$

Heaviside step-function $\theta(t)$ defined: $\begin{cases} \theta(t<0) = 0 \\ \theta(t \geq 0) = 1 \end{cases}$

$$G(\tau) = \omega_p^2 \, e^{-\gamma\tau/2} \frac{\sin \omega_R \tau}{\omega_R} \, \theta(\tau)$$

Jump → 13

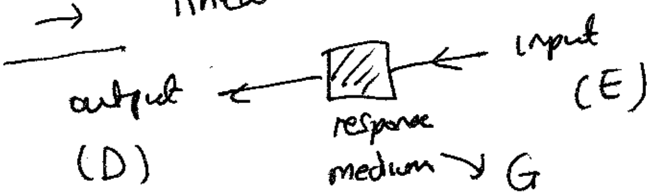
Green's function for damped SHO.

$$\frac{1}{\epsilon_0} \vec{D}(\vec{x}, t) = \vec{E}(\vec{x}, t) + \int_0^\infty G(\tau) \vec{E}(\vec{x}, t-\tau) \, d\tau$$

$\theta(\tau)$ in $G(\tau)$ no contribution $\tau < 0$ (∵ causality - otherwise $\vec{E}(\vec{x}, t')$ $t' > t$ would 'influence' $\vec{E}(\vec{x}, t)$!

→ Causality

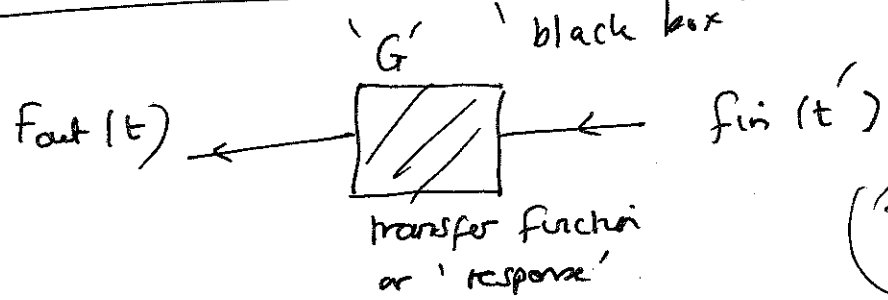
→ linear response.



$G(\tau)$ ~ exponentially decays $\tau > 1/\gamma$ - most 'influence' occurs $0 < \tau \lesssim 1/\gamma$

↳ Skip to here for more succint approach to K-K relation.

linear response & Causality:



$$F_{out}(\omega) = G(\omega)\, f_{in}(\omega)$$

Causality $\Rightarrow$ $t' \leq t$ – i.e. can't have an output at time t, before the time t' of input!

$$F_{out}(t) = \int_{-\infty}^{\infty} G(\tau)\, F_{in}(\underbrace{t-\tau}_{= t'})\, d\tau$$

and $G(\tau) = 0$ for $\tau < 0$.

causality:

$$\boxed{G(t) = G(t)\, \theta(t)}$$

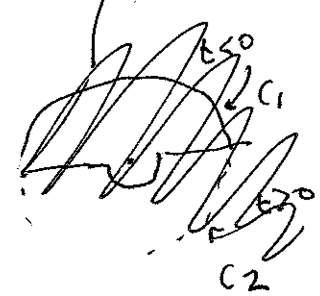
what is F.T. of $\theta(t)$? — tricky! answer is

$$\theta(\omega) = \int_{-\infty}^{\infty} \theta(t)\, e^{i\omega t}\, dt = \left[\pi\delta(\omega) + \frac{i}{\omega}\right]$$

understood as distribution

$$\left[\!\left[\ \text{check: } \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta(\omega)\, e^{-i\omega t}\, d\omega = \theta(t)\ \right]\!\right]$$

$$\delta \to 0 \quad \int_{-\infty}^{0+\delta} \frac{e^{-i\omega t}}{\omega}\, d\omega + \int_{\delta}^{\infty} \frac{e^{i\omega t}}{\omega}\, d\omega$$



Then $G(\omega) = \int_{-\infty}^{\infty} G(t)\, \theta(t)\, e^{i\omega t}\, dt$

convolution theorem in ω-space

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')\, \theta(\omega-\omega')\, d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega')\left(\pi\,\delta(\omega-\omega') + \frac{i}{\omega-\omega'}\right)$$

$$G(\omega) = \frac{1}{2}G(\omega) + \frac{i}{2\pi} P \int_{-\infty}^{\infty} \frac{G(\omega')}{\omega-\omega'}\, d\omega'$$

$$\therefore \quad \boxed{G(\omega) = \frac{-i}{\pi} P \int_{-\infty}^{\infty} \frac{G(\omega')}{\omega'-\omega}\, d\omega'}$$

P indicates principle value. This assumes $G(\omega') \to 0$ at least as fast as $\frac{1}{|\omega'|}$ as $|\omega'| \to \infty$; and that it is analytic in upper half complex plane

This relation holds a lot of physics!

$G(\omega) = \mathrm{Re}[G(\omega)] + i\, \mathrm{Im}[G(\omega)]$

$$\boxed{\begin{aligned} \mathrm{Re}[G(\omega)] &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}[G(\omega')]}{\omega'-\omega}\, d\omega' \\ \mathrm{Im}[G(\omega)] &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}[G(\omega')]}{\omega'-\omega} \end{aligned}}$$

So $\mathrm{Re}\,G$, $\mathrm{Im}\,G$ are not independent!

$\mathrm{Re}\,G \leftrightarrow \mathrm{Im}\,G$

$\to$ just causality!!

- Quite general result in linear response theory.

Back to our setup. $\vec{P}(\vec{r}, \omega) = \chi_e(\omega) E(\omega, \vec{r})$

output, 'transfer', input

Since $\chi_e(\omega) \xrightarrow{|\omega| \to \infty} 0$ [in our model
$\chi_e(\omega) = \frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{e^2}{m(\omega^2 - \omega_0^2 - i\omega\gamma)}$
$|\chi| \xrightarrow{|\omega| \to \infty} 0 \sim 1/|\omega|^2$].

$\chi_e(\omega)$ will satisfy these constraints on real, img. parts $\mathrm{Re}\,\chi_e$, $\mathrm{Im}\,\chi_e$.

The famous Kramers-Kronig relations are basically just this but instead of $\chi_e(\omega)$ we express things in terms of permittivity $\epsilon(\omega)$

$$\chi_e(\omega) = \frac{1}{\epsilon_0} \epsilon(\omega) - 1 = (\epsilon_r(\omega) - 1)$$

$\therefore |\omega| \to \infty$; $\epsilon_r(\omega) \to 1$ not 0. So

$$\boxed{\begin{aligned} \mathrm{Re}[\epsilon_R(\omega)] &= 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}[\epsilon_r(\omega')]}{\omega' - \omega} d\omega' \\ \mathrm{Im}[\epsilon_r(\omega)] &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}[\epsilon_r(\omega') - 1]}{\omega' - \omega} d\omega' \end{aligned}}$$

$\longmapsto$ 16'

$$\epsilon_r(\omega) = 1 + \int_0^\infty G(\tau)\, e^{i\omega\tau}\, d\tau$$

Some properties:-

1) $\boxed{\epsilon_r(-\omega) = [\epsilon_r^*(\omega^*)]^*}$; $G^*(\tau) = G(\tau)$ (real fctn)

$\Rightarrow$ $\epsilon_r(\omega)$ is an analytic function in upper ½ ω-complex plane [basically means $\epsilon_r = \epsilon_r(e^{i\omega})$]

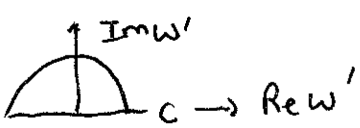
Cauchy's Theorem tells us that for complex analytic functions $f(z)$ for any point z inside closed contour C,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')\, dz'}{z' - z}$$

In our case

$$\epsilon_r(z) = 1 + \frac{1}{2\pi i} \oint_C \frac{\epsilon_r(\omega') - 1}{\omega' - z}\, d\omega'$$

with $\operatorname{Im} z > 0$ and C is usual semi-circular contour



– no contribution from arc as $\to \infty$ because $\epsilon_r \to 1$ for $\operatorname{Im} \omega' \to \infty$

$$\therefore\ \epsilon_r(z) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[\epsilon_r(\omega') - 1]}{\omega' - z}\, d\omega'$$

Now take $z=\omega+i\epsilon$, $\epsilon>0$.

$$\epsilon_r(\omega) = \lim_{\epsilon\to 0}\left[1+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\epsilon_r(\omega')-1}{\omega'-\omega-i\epsilon}\,d\omega'\right]$$

simple pole $\omega'=\omega$

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0-i\epsilon}\,dx = P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0} + i\pi f(x_0)$$

P – principal value

$$\epsilon_r(\omega) = 1+\frac{1}{\pi i}\,P\int_{-\infty}^{\infty}\frac{(\epsilon_r(\omega')-1)\,d\omega'}{\omega'-\omega}$$

$\Rightarrow$

$$\boxed{\begin{aligned}\mathrm{Re}(\epsilon_r(\omega)) &= 1+\frac{1}{\pi}\,P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\epsilon_r(\omega')}{\omega'-\omega}\,d\omega' \\ \mathrm{Im}(\epsilon_r(\omega)) &= -\frac{1}{\pi}\,P\int_{-\infty}^{\infty}\left[\frac{\mathrm{Re}(\epsilon_r(\omega'))-1}{\omega'-\omega}\right]d\omega'\end{aligned}}$$

~~We can write this in a different way~~

~~by analyticity;~~ $\cancel{\epsilon_R(-\omega) = [\epsilon_R(\omega^*)]^* \Rightarrow \mathrm{Re}\,\epsilon_R(-\omega) = \mathrm{Re}(\epsilon_R(\omega))}$

$\cancel{\mathrm{Im}\,\epsilon_R(-\omega) = -\mathrm{Im}(\epsilon_R(\omega))}$

$\cancel{\therefore\ \mathrm{Re}(\epsilon_R(\omega)) = \frac{1}{2}[\mathrm{Re}\,\epsilon_R(\omega) + \mathrm{Re}\,\epsilon_R(-\omega)]}$

$\cancel{\mathrm{Im}(\epsilon_R(\omega)) = \frac{1}{2}[\mathrm{Im}\,\epsilon_R(\omega) - \mathrm{Im}\,\epsilon_R(-\omega)]}$

K-k and dispersion

Going back to plane waves: In a medium

$\epsilon_0, \mu_0 \leftrightarrow \epsilon(\omega), \mu(\omega)$.

phase velocity $= \frac{\omega}{\tilde{k}(\omega)} = \frac{1}{\sqrt{\mu\epsilon}} = c/\tilde{n}$ complex refractive index.

$\tilde{n} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$. But now clearly $n = n(\omega)$

$\Rightarrow$ dispersion and phase velocity depends on ω, $= \frac{c}{n(\omega)}$.

plane wave: $\vec{E}(t,z) \sim \vec{E}_0 \, e^{i\tilde{k}(\omega) z - \omega t}$ (complex wave number.)

dispersion relation (from satisfying wave equation)

$\cancel{k^2 = \mu\epsilon\omega^2, \; k = \omega\sqrt{\mu\epsilon} = k(\omega)}$).

$\tilde{k}(\omega) = \tilde{n}(\omega)\frac{\omega}{c}$

$\vec{E}(\omega, z) = \vec{E}_0 \, e^{i\tilde{k}(\omega) z}$ $\quad (\vec{E}_{phys} = Re(\vec{E}\cdot))$

Now $k(\omega)$ is complex in general (since we found $\epsilon(\omega)$ is complex). $\therefore n(\omega)$ complex;

$\tilde{n}(\omega) = n(\omega) + i\kappa(\omega)$ — kappa — extinction coeff.

$n(\omega)$: real refractive index

$$\Rightarrow \tilde{k}(\omega) = \underbrace{\beta(\omega)}_{= n(\omega)\omega/c} + i\underbrace{\frac{\alpha(\omega)}{2}}_{= \kappa(\omega)\omega/c}$$

$\alpha(\omega)$ - absorption coeff.

$\vec{E}_{phys} = Re\left(\vec{E}_0 \, e^{i\beta(\omega) z} \, e^{i\omega t} \, e^{-\frac{\alpha}{2} z}\right)$ decay of wave