ELECTRO - MAGNETIC THEORY

(4261)

Units essentially S.I.

- 2 most important constants we will use throughout course:

$$\mu_0 = 4\pi \times 10^{-7} \; MLT^{-2} I^{-1}$$

$$\epsilon_0 = \frac{1}{\mu_0 c^2} = 8.854 \times 10^{-12} \; M L^{3} T^{-4} I^{-2}.$$

ϵ_0 = permittivity of free space (= vacuum)

μ_0 = magnetic permiability " " " "

[[Note: here vacuum means no 'medium'. Can still have sources, currents as we will discuss]]

While ϵ_0, μ_0 are constants in vacua, their values in different media can be different and also need not be constant e.g. as we shall see later ϵ can depend on the frequency of a wave moving in a medium etc]]

Before discussing fundamental equations describing electric and magnetic fields (Maxwell's equations) lets recap on some basic formulae in vector calculus that we'll be using throughout course.

Scalar, Vector fields

$\phi(x,y,z,t)$ ↑ scalar field — 3d co-ords x,y,z (cartesian mostly) — time t.

at each point in space, time there is a real number $= \phi(x,y,z,t)$.

Similarly vector field $\vec{V}(x,y,z,t)$ defines a vector at every point of S-T. $\underbrace{(V_x, V_y, V_z)}_{\text{components}}$

Vector valued operators:-

gradient 'grad' $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, in Cartesian.

e.g. $\underbrace{\vec{\nabla}\phi(x,y,z,t)}_{\text{vector field}} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$

Divergence of vector field $\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$

<u>Curl:</u>

$$\text{curl}(\vec{V}) \equiv \vec{\nabla} \times \vec{V} \quad \text{— result is a vector field:}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} , \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} , \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right).$$

Component notation: $(x, y, z) = x^i \quad i = 1, 2, 3 \quad \left(\begin{matrix} x^1 = x \\ x^2 = y \\ x^3 = z \end{matrix} \right)$

Similarly $\vec{V} = \{ V_i \} \quad i = 1, 2, 3$

$$\leftrightarrow (V_x, V_y, V_z).$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \leftrightarrow \left\{ \frac{\partial}{\partial x^i} \right\}, \quad i = 1, 2, 3$$

Then $(\vec{\nabla}\phi)_i = \dfrac{\partial \phi}{\partial x^i}$; $\vec{\nabla} \cdot \vec{V} = \displaystyle\sum_{i=1}^{3} \frac{\partial}{\partial x^i} V_i$

Use Einstein summation convention (here in 3d – later we will discuss space-time)

$$\sum_{i=1}^{3} A_i B_i = A_i B_i \quad (\text{supress } \textstyle\sum_i \text{ although it's there!})$$

So $\vec{\nabla} \cdot \vec{V} = \left(\dfrac{\partial}{\partial x^i} V_i \right)$

$$(\vec{\nabla} \times \vec{V})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} V_k$$

ϵ_{ijk} – totally anti symmetric symbol.

$\epsilon_{123} = +1$,

$\epsilon_{ijk} = -\epsilon_{jik}$

$$(\nabla \times V)_1 = \epsilon_{123} \frac{\partial}{\partial x^2} V_3 + \epsilon_{132} \frac{\partial}{\partial x^3} V_2$$

$$= \epsilon_{123} (\partial_2 V_3 - \partial_3 V_2) = (\partial_y V_z - \partial_z V_y) + 2 \text{ other ident } i \leftrightarrow k,\ j \leftrightarrow k$$

Identities:

div curl $\vec{V} = 0$ for any $\vec{V}$.

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0 .$$

Use components, and previous definition of $\vec{\nabla}\times\vec{V}$:-

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = \frac{\partial}{\partial x^i}(\vec{\nabla}\times\vec{V})_i$$

$$= \frac{\partial}{\partial x^i}\,\epsilon_{ijk}\left(\frac{\partial}{\partial x^j}V_k\right)$$

But ϵ_{ijk} is antisymmetric in any two indices.

while obviously $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}V_k = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}V_k$ is symmetric

in i,j $\quad\therefore$ rhs $\underline{= 0}$.

curl grad $\phi = 0$

$$\left[\vec{\nabla}\times(\vec{\nabla}\phi)\right]_i = \epsilon_{ijk}\frac{\partial}{\partial x^j}(\vec{\nabla}\phi)_k$$

$$= \epsilon_{ijk}\frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^k}\phi\right)$$

$= 0$ by same argument as above.

product rules:

$$\vec{\nabla}\cdot(\phi\vec{V}) = (\vec{\nabla}\phi)\cdot\vec{V} + \phi\,\vec{\nabla}\cdot\vec{V}$$

$$\vec{\nabla}\times(\phi\vec{V}) = (\vec{\nabla}\phi)\times\vec{V} + \phi(\vec{\nabla}\times\vec{V}).$$

- so both ~~[illegible]~~ div and curl operation act like derivations - i.e. satisfy Liebniz rule

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2\vec{V}$$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}$$

[ex: prove use identity

$$\epsilon_{ijk}\,\epsilon_{ilm} = (\delta_{jl}\,\delta_{km} - \delta_{kl}\,\delta_{jm})$$

$$\delta_{ij} = \begin{cases} +1, & i=j \\ 0 & i\neq j \end{cases}$$

]

Indeed for any $\vec{A}, \vec{B}, \vec{C}$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Integrals involving Scalar, Vector fields

closed region of volume V
boundary S.

$d\vec{s} = ds\,\vec{n}$; $\vec{n}$ unit vector $\perp$ surface at given point.

ds — infinitesimal area

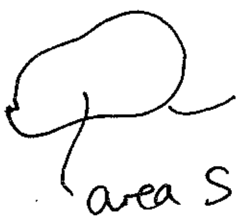
Gauss' law:

$$\oint_S \vec{A}\cdot d\vec{s} = \oint_S \vec{A}\cdot\vec{n}\,ds = \int_V \vec{\nabla}\cdot\vec{A}\,dV$$

dV = infinitesimal volume element ($= dx\,dy\,dz$ cartesians)

Above holds for any vector field $\vec{A}$.

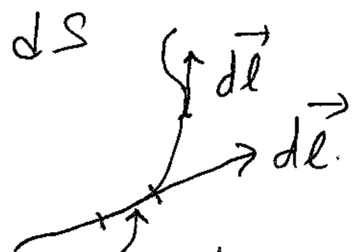
Stoke's Theorem



Now consider surface S with closed boundary given by curve C (loop)

$$\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \int_S (\vec{\nabla}\times\vec{A})\cdot\vec{n}\,ds$$

$$= \oint_C \vec{A}\cdot d\vec{l}$$



$dl = |d\vec{l}|$
direction tgt C.

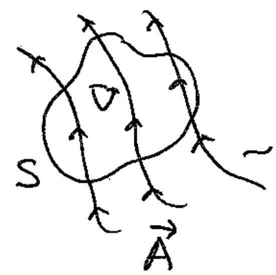
geometric Interpretation:

Both Gauss Law and Stoke's Theorem have nice interpretation in terms of the geometry of vector fields in certain instances.

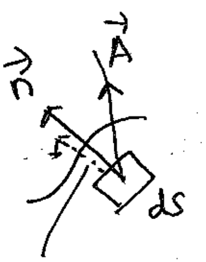
For Example suppose in volume V, $\vec{\nabla}\cdot\vec{A} = 0$

$$\Rightarrow \oint_S \vec{A}\cdot d\vec{s} = 0 \;.$$

$$= \oint_S (\vec{A}\cdot\vec{n})\, ds$$



some lines whose ~~tangent~~ vector field $\vec{A}$



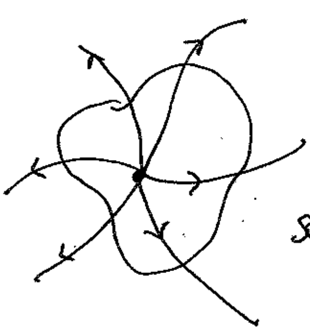
$(\vec{A}\cdot\vec{n})$ = projection of $\vec{A}$ onto $\vec{n}$.

$=$ total flux of $\vec{A}$ through surface $S = 0$.

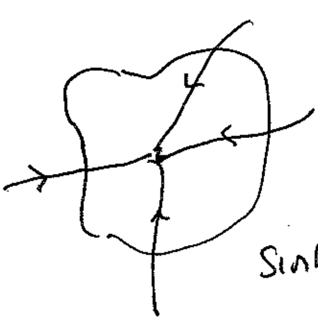
$\Leftrightarrow$ number of field lines of $\vec{A}$ entering S

$=$ " " " leaving S

$\Leftrightarrow$ no 'sources' or 'sinks' of $\vec{A}$.



source of $\vec{A}$



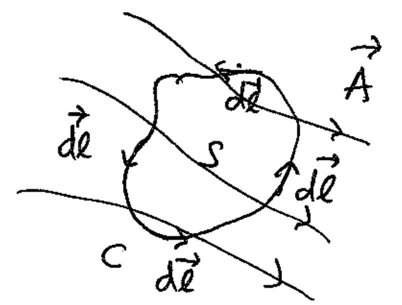
sink of $\vec{A}$

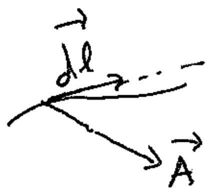
both cases $\oint \vec{A}\cdot d\vec{s} \neq 0$.

$\Rightarrow$ sources / sinks are responsible for $\vec{\nabla}\cdot\vec{A} \neq 0$ in some region.

Similarly if now we imagine there is some surface S, with boundary given by closed curve C, and on S, $\vec{\nabla} \times \vec{A} = 0$.

Stoke's theorem $= \oint_C \vec{A} \cdot \vec{dl} = 0$

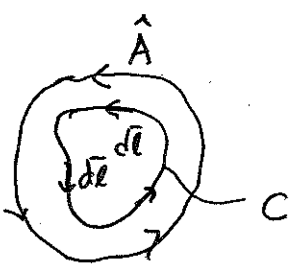




$\vec{A} \cdot \vec{dl}$ = projection of $\vec{A}$ onto $\vec{dl}$.

As you go around loop C, clear that adding $\vec{A} \cdot \vec{dl}$ $\Leftrightarrow$ evaluating 'circulation' of $\vec{A}$ around C.

no net circulation $\Leftrightarrow$ no closed loops of $\vec{A}$.



– now should be clear $\oint \vec{A} \cdot \vec{dl} \neq 0$ because at each point on C, $(\vec{A} \cdot \vec{dl}) > 0$.

Both of these geometrical interpretations will be important when we consider $\vec{E}$, $\vec{B}$ fields later.

Maxwell's Equations (in vacua)

Before considering these, lets first think about what information is needed to determine a vector field $\vec{A}$ in 3d. In fact modulo a few subtleties it is enough to determine $\vec{\nabla}\cdot\vec{A}$ and $\vec{\nabla}\times\vec{A}$.

$\vec{A} = \vec{A}_d + \vec{A}_c$ where A_d is divergence-free, $\boxed{\vec{\nabla}\cdot\vec{A}_d = 0}$ and $\vec{A}_c$ is curl free, $\boxed{\vec{\nabla}\times\vec{A}_c = 0}$

Then clearly $(\vec{\nabla}\cdot\vec{A}) = \vec{\nabla}\cdot\vec{A}_c$; $(\vec{\nabla}\times\vec{A}) = (\vec{\nabla}\times\vec{A}_d)$ and so solving last 2 Eqs $\Rightarrow$ knowledge of $\vec{\nabla}\cdot\vec{A}$, $\vec{\nabla}\times\vec{A}$ is enough to determine $\vec{A}_d$, $\vec{A}_c$ and hence $\vec{A}$. Subtlety is $\vec{A}$ only determined up to $\vec{\nabla}\phi$, $\nabla^2\phi = 0$. But b.c's / regions in space where $\vec{A}$ defined can remove this ambig.

Thus in trying to find physical Equations that $\vec{E}$, $\vec{B}$ should satisfy it is enough to specify 4 equations: for $\vec{\nabla}\cdot\vec{E}$, $\vec{\nabla}\cdot\vec{B}$, $\vec{\nabla}\times\vec{E}$ and $\vec{\nabla}\times\vec{B}$

$\vec{E} = -\vec{\nabla}\phi$ — ϕ electric potential.

— this is reasonable — and is in complete analogy with gravitational force (Newton) $\vec{F}_{grav} = -\vec{\nabla}\Phi_{grav}$

So force on a unit electric charge $\vec{F} = \vec{E}$ ($\vec{F} = q\vec{E}$)

$\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = -\nabla^2\phi$; ~~for gravity~~

$= 0$ in vac.

$$\boxed{\vec{\nabla}\cdot\vec{E} = 0}$$

~~more generally~~

Similarly for $\vec{B}$, $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ in vacuum.

If either $\vec{\nabla}\cdot\vec{E}$, $\vec{\nabla}\cdot\vec{B}$ were $\neq 0$, the rhs would be related to sources (i.e. electric or magnetic charge) — but in vacua these are zero.

Now what about $\vec{\nabla}\times\vec{E}$, $\vec{\nabla}\times\vec{B}$?

Faraday :— discovered electromagnetic induction — $\vec{B}$ field changing in time induces $\vec{E}$ field.

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Magnetic analogue:

$$\boxed{\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}.$$

$\uparrow$ $[c] = LT^{-1}$.

looking at $[E], [B]$ — see need a constant $[\;] = L^{-2}T^{2}$

Can test latter experimentally [Kohlrausch + Weber 1849]

and find $c = 3.1 \times 10^8 \, m s^{-1}$.

– and experiment by Fizeau and others around this time had found velocity of light was also close to same value!

This was no accident as Maxwell in his ingenuity showed as :–

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\Rightarrow \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}.$$

But $= -\nabla^2 \vec{E}$ (see earlier)

$$\therefore \quad \boxed{\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}}$$ – wave equation!

Similarly one finds for $\vec{B}$:

$$\boxed{\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}}$$ – another wave equation!

$\therefore$ c must correspond to a wave velocity. Maxwell concluded that light waves must consist of electric and magnetic waves travelling at $3 \times 10^8 \, m s^{-1}$. All this is in vacua – no need for Aether!

Sources

Now let's introduce sources for $\vec{E}, \vec{B}$.

First imagine we introduce a static electric charge density $\rho(\vec{x})$ ($\frac{d\rho}{dt} = 0$). This will now modify rhs of $\vec{\nabla}\cdot\vec{E} = 0$:-

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho}$$

where we have introduced a constant, ϵ_0, called electric permittivity of free space. It has dimensions

since $[E] = [\text{Force}]/[\text{charge}] = N/C$; $[\rho] = C/L^3$

$\Rightarrow$ $[\epsilon_0] = 8.85\times10^{-12}\, C^2 N^{-1} m^{-2}$ (measured expmt.)

Why is this correct eq^n? $\vec{E} = -\vec{\nabla}\phi$

$$\vec{\nabla}\cdot(-\vec{\nabla}\phi) = -\nabla^2\Phi = \frac{1}{\epsilon_0}\rho(x).$$

~~Now consider the potential produced by point charge located at $\vec{x} = \vec{x}_0$~~

$$\cancel{\rho(\vec{x}-\vec{x}_0) = Q\,\delta^{(3)}(\vec{x}-\vec{x}_0)}$$

~~- Simpler - use spherical polar co-ords:~~

Solution can be expressed :-

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{|\vec{r}-\vec{r}'|}\,dV$$

Later when we discuss Green's functions we will prove this statement. For now we can motivate it in following way.

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^{(3)}(\vec{r}) \quad ; \quad r = |\vec{r}\,|$$

and $\delta^{(3)}$ is the 3d Dirac delta function. (more later)

For $r \neq 0$ $\quad \delta^{(3)}(\vec{r}) = 0$ as is clear LHS

using $\left(\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \text{terms include } \frac{\partial}{\partial\theta}, \partial/\partial\phi\right)$

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\left(\frac{1}{r}\right) = 0 \quad \text{if we have } r \neq 0$$

On other hand, integrating Eqn over a unit 3 sphere centred at $r=0$.

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV \overset{\text{Gauss's Law}}{=} \int_{S^2} \left(\vec{\nabla}\left(\frac{1}{r}\right)\right)\cdot d\vec{S} = -\int_{S^2} \frac{1}{r^3}\,\vec{r}\cdot d\vec{S}$$

unit 2-sphere $\to S^2$

$$= -\int_{r=1} \frac{1}{\cancel{r^2}}\,\cancel{r^2}\,d\Omega = -4\pi$$

$$= \int_V (-4\pi)\,\delta^3(r) = -4\pi \underbrace{\int \delta^{(3)}(r)\,dV}_{=1}$$

So LHS of Eqn has properties of rhs. So $-\frac{1}{4\pi r}$ 'inverse' of $-\nabla^2$

- See later for more rigorous proof.

Furthermore our general solution reduces to well known expression for Φ due to a point charge. Take $\rho(\vec{r}\,') = Q\,\delta^{(3)}(\vec{r}\,')$

$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{Q\,\delta^{(3)}(\vec{r}\,')\,dV}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad ; \quad \vec{E} = -\vec{\nabla}\Phi = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}$$

$$\int_V \nabla\cdot E\, dV = \oint_S E\cdot dS = \frac{1}{\epsilon_0}\int \rho\, dV = Q/\epsilon_0$$

What about a 'magneto-static' charge density?

This would modify $\vec{\nabla}\cdot\vec{B} \neq 0$. (~ magnetic charge density)

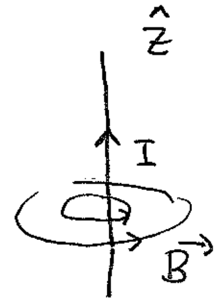
Well in principle can (should?) exist but so far no magnetic monopoles have ever been observed!
(one of most important puzzles of recent times...)

So we conclude that $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

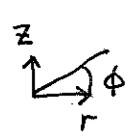
Currents

So what 'creates' the field $\vec{B}$ if not magnetic charges?

: electric currents. Biot-Savart



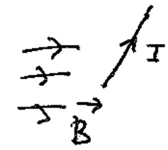
$$\vec{B}(\rho) = \frac{\mu_0 I}{2\pi\rho}\,\hat{\phi}$$

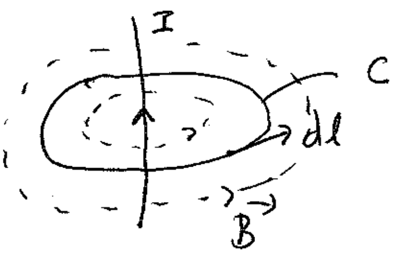


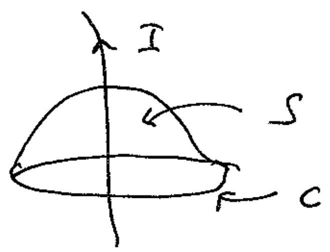
cyl. co-ords

μ_0 is a fundamental constant relating strength of $\vec{B}$ field and electric current I.

$\mu_0 = 1.257 \times 10^{-6}\ N A^{-2}$

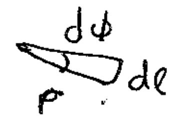
[analog of $[\vec{E}]$ = Force/unit charge
$[\vec{B}]$ = Force/unit current/unit length of wire ]





Consider integrating $(\vec{\nabla}\times\vec{B})$ over surface S bounded by arb. closed curve C enclosing current carrying wire.

$$\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{S} \overset{\text{Stoke's}}{=} \oint_C \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I$$



$dl = r\,d\phi$

If $\vec{J}$ = electric current density
= current/unit area on S

$$\mu_0 I = \mu_0 \int_S \vec{J}\cdot d\vec{S}$$

$\therefore$ $\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}}$ $\Longrightarrow$ Ampere's Law

What about $\vec{\nabla}\times\vec{E}$? Since there are no magnetic charge carriers analagous to electric charges, there are no 'magnetic' currents.

Thus $\vec{\nabla}\times\vec{E} = 0$ in static case.

So, relaxing condition that $\vec{E}, \vec{B}$ are time-independent complete set of Maxwell Equations in free space with sources :-

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\times\vec{B} = +\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ why?

charge conservation.

Charge Conservation.

Take div. of $\vec{\nabla}\times\vec{B}$ Eq^n:

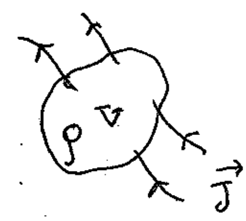
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 = \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) + \mu_0 \vec{\nabla}\cdot\vec{J}$$

($\vec{\nabla}\cdot\vec{B} = 0$)

But $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ $\therefore$ $\mu_0\epsilon_0 \frac{\partial}{\partial t}\left(\rho/\epsilon_0\right) + \mu_0 \vec{\nabla}\cdot\vec{J} = 0$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$ — conservation of charge

$$\int_V (\vec{\nabla}\cdot\vec{J})\,dV = \int_S \vec{J}\cdot d\vec{S} = \frac{\partial}{\partial t}\int_V \rho\, dV = \frac{\partial Q}{\partial t}$$



So rate of flow of charge through S is related to the 'flux' of electric current passing through it.

This also explains why $\mu_0 \vec{J}$ is present in Maxwell's Eqn.

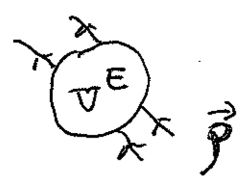
- it is needed for charge conservation.

Energy - Momentum associated with $\vec{E}, \vec{B}$

- Will become more clearer later in discussing Lorentz invariance and Lagrangian formulation of Maxwell's Equations why there should be a momentum associated with $\vec{E}, \vec{B}$

For now we can motivate it as follows.

Energy (like charge) must also have an eqn describing it's microscopic conservation, that is $\frac{\partial E_V}{\partial t}$ - rate of change of energy in volume V ~~density~~ must be proportional to ~~momentum~~ flux of an current $\vec{P}$, through a surface.



$$\Rightarrow \quad \frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla}\cdot\vec{P} = 0 . \qquad \left(\frac{\partial}{\partial t}\int \mathcal{E}\, dV = \oint_S \vec{P}\cdot d\vec{S} \right)$$

$\mathcal{E}$ = energy density

$(\leftrightarrow \rho)$

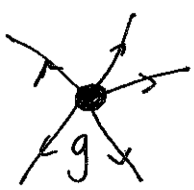
We will see later that infact :-

$$\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \qquad E^2 = \vec{E}\cdot\vec{E} \ ; \ B^2 = \vec{B}\cdot\vec{B}$$

Using Maxwell's Equation (Ex.) one can then prove that $\boxed{\vec{P} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})}$ - so called Poynting's Vector - 'direction of energy flow'

~~Note that $\vec{\Phi}, \vec{E}, \vec{B}$~~

What if magnetic monopoles exist - physical implications ?



$$\oint_S B\cdot ds = g\mu_0 \qquad \left(\text{c.f. } \frac{\Phi}{\varepsilon_0} \text{ for electric charge}\right)$$

$$\Rightarrow \vec{B}(r) = \frac{g\mu_0}{4\pi r^3}\vec{r}.$$

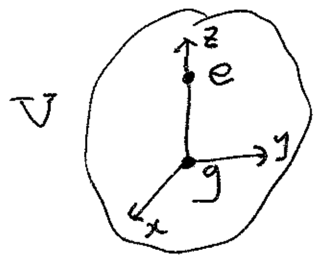
Big suprise : g cannot take on any value - it is quantized in units of $1/e$!

$$ge = \frac{1}{2} n\hbar c$$

or vice versa - if a single magnetic monopole exists some where in Universe, electric charge is quantized for $e = \frac{1}{2}\frac{n\hbar c}{g}$

various ways of deriving quantization condition [Dirac was first to derive this famous result]. Simplest is to consider

A.M ~~density~~ due to e and g charge e.g along z axis:-



A.M. density

$$L_z = \int_{dV} (\vec{r} \times \vec{P}_{e.m})_z \, dV = -eg/c \qquad \left(\vec{P}_{e.m} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \right)$$

Q.M. $\Rightarrow$ L_z can take on values $\left(\frac{n}{2}\right)\hbar$ $n = 0, 1 ..$

$\Rightarrow$ quantization condition.

$$g^2/e^2 \gtrsim \frac{1}{4}\left(\hbar^2 c^2 / e^4\right) \gtrsim \frac{1}{4}\left(137\right)^2$$

(fine structure constant Experimental $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ unit where $(4\pi\varepsilon_0 = 1)$)

$\therefore$ $\underline{g^2 \simeq 5000 \times e^2}$

So force e.g between 2 magnetic monopoles of charge g is about 5000 larger than between 2 electric charges having same separation!