Weeks 10 & 11 : Lagrangian formulation of Electrodynamics

Lagrangian of charged particle:

Since we know Lagrangians, action principle ⟹ Newton's laws of motion eg for a particle in a potential, natural to ask how this generalizes to case of a point charged particle in presence of $\vec{E}, \vec{B}$ fields? Also, since we have discovered that Maxwell's Equations have a Lorentz covariant formulation - such a Lagrangian (and hence action) must be Lorentz invariant.

Recall particle equations of motion:-

$$\frac{dU^\alpha}{d\tau} = \frac{q}{m} F^{\alpha\beta} U_\beta$$

$U^\alpha = (c, \vec{u})$ is particles 4-velocity, $F^{\alpha\beta}$ the Maxwell field strength and τ proper time

Action principle says these can be derived as Euler-Lagrange Equation that follow from stationary of action $A = \int_{t_1}^{t_2} dt\, L(\vec{r}(t), \dot{\vec{r}}(t))$; $\delta A = 0$.

Standard manipulations give:-

$$\delta A = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \vec{r}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\right) \right] \cdot \delta \vec{r} \, dt = 0$$

$\Rightarrow$ Euler Lagrange eqs: (E-L) $\boxed{\frac{\partial L}{\partial \vec{r}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\right) = 0}$

or $\frac{d\vec{P}}{dt} = \frac{\partial L}{\partial \vec{r}}$ with $\vec{P} \equiv \left(\frac{\partial L}{\partial \dot{\vec{r}}}\right)$.

$L = T - V = $ k.e $-$ p.e $\Rightarrow \frac{\partial L}{\partial \vec{r}} = -\vec{\nabla} V = \vec{F}$ (force)

So E-L Eqs $\leftrightarrow$ Newton's Eqs. of motion.

What about a relativistic point particle?

Well $A = \int_{t_1}^{t_2} dt\, L = \int_{\tau_1}^{\tau_2} d\tau \left(\frac{dt}{d\tau}\right) L = \int_{\tau_1}^{\tau_2} d\tau (\gamma L)$

We have seen that τ is a Lorentz Scalar - thus requiring A to be a Lorentz invariant quantity $\Rightarrow$

γL must be a <u>scalar</u> wrt Lorentz transformations.

Consider a free particle - then $V=0$ and so L independent of $\vec{r}$. (only depends on $\dot{\vec{r}}$) $\therefore$ $L = L(\underbrace{U^\alpha U_\alpha}_{\text{Lorentz invariant combination.}})$

But $U^\alpha U_\alpha = c^2$! $\Rightarrow$ γL_{free} = constant.

$$L_{free} = \frac{-mc^2}{\gamma(\vec{u})} = -mc^2\sqrt{1-\beta^2} \qquad \begin{aligned}\beta^2 &= \vec{\beta}\cdot\vec{\beta} \\ \vec{\beta} &= \vec{u}/c = \dot{\vec{r}}/c\end{aligned}$$

Where constant is in fact $\underline{\underline{-mc^2}}$, in order to reproduce the familiar expression $L_{free} \approx \frac{1}{2}mu^2$ in the non-relativistic limit.

The momentum $\vec{p}$ conjugate to $\dot{\vec{r}}$ is

$$\vec{p} = \frac{\partial L}{\partial \vec{u}} = m\gamma\vec{u}$$ – just relativistic 3-momentum.

Eq. motion is $\boxed{\frac{d\vec{P}}{dt} = 0}$

Charged particle:

If we first consider a stationary charge, $V = q\Phi = qA_0c$ where Φ = electric potential (P.E/unit charge). For a charge that is also moving, it produces a current $\vec{J} \sim q\vec{u}$. We have seen that in a Lorentz covariant formulation $\vec{J} \to J^\mu = qU^\mu$

Hence we can guess that $qA_0c \to$ $$J^\mu A_\mu = qcA_0\gamma - q\vec{u}\cdot\vec{A}\gamma$$

So $\gamma L = -mc^2 - J^\mu A_\mu$

$$\boxed{L = -mc^2\sqrt{1-\vec{u}\cdot\vec{u}/c^2} - qcA^0 + q\vec{u}\cdot\vec{A}}$$

Then the conjugate momentum is

$$\underline{\vec{P}} \equiv \frac{\partial L}{\partial \dot{\vec{r}}} = \frac{\partial L}{\partial \vec{u}} = \underbrace{\vec{p}}_{m\gamma\vec{u}} + q\vec{A}$$

shifted by $\vec{A}$.

$m\gamma\vec{u}$ – mechanical momentum of particle.

One can check that E–L Equation of motion are now:–

$$\underbrace{\frac{d}{dt}p^i + q\left(\frac{\partial A^i}{\partial t} + \sum_j \frac{\partial A^i}{\partial x^j}\frac{\partial x^j}{\partial t}\right)}_{\frac{dP^i}{dt}} = \underbrace{-qc\frac{\partial A^0}{\partial x^i} + q\sum_j u^j \frac{\partial A^j}{\partial x^i}}_{\frac{\partial L}{\partial x^i}}$$

But these Equations are exactly:–

$$\dot{\vec{p}} = q\left[\vec{E} + \vec{u}\times\vec{B}\right]$$ derived in earlier lectures.

where $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$; $\vec{B} = \vec{\nabla}\times\vec{A}$

– So this is a consistency check that we have found correct expression for charged particle action.

Likewise one can easily derive Hamiltonian

$$H = \underbrace{\left[\vec{p}\cdot\vec{p}\,c^2 + m^2c^4\right]^{\frac{1}{2}}}_{\substack{\text{Standard relativistic energy} \\ E}} + \underbrace{qcA_0}_{\text{P.E due to } \vec{E}\text{-field}}$$

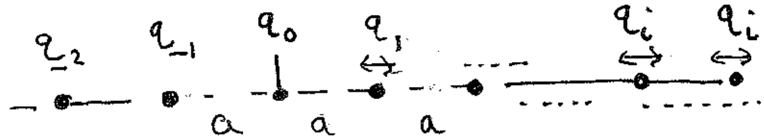
Lagrangian for the Electro-Magnetic field

Thus far in your studies you have likely only considered action principle ('principle of least action') as applying to a system of point particles. That is the Lagrangian $L = L(q^i(t), \dot{q}^i(t))$ where $q^i(t), \dot{q}^i(t)$ represent position & velocities of i^{th} particle and $i = 1, 2, \ldots N$.

This is a system with finite number of degrees of freedom ($2N$, N positions, N velocities).

Imagine now that our dynamical system contained an infinite number of degrees of freedom e.g. consider series of atoms fixed at 'lattice points' $x^i = ia$

In e.g. 1d:-



$i = 0, \pm 1, \pm 2, \pm 3, \ldots$

If we call q_i the <u>fluctuation</u> of i^{th} atom about it's mean position $x^i = ia$, we can write total K.E. of system as:-

$$T = \sum_{i=-\infty}^{\infty} \frac{1}{2} m (\dot{q}_i)^2$$

$$V = \sum_{i=-\infty}^{\infty} \frac{1}{2} k \underbrace{(q_{i+1} - q_i)}^2 + U(q_i)$$

where k is some constant and $U(q_i)$ is 'self-energy' of i^{th} atom. The term $(q_{i+1}-q_i)^2$ is an 'elastic' type term [e.g. consider elastic media Hookes Law Force ~ displacement ⇒ Elastic P.E ~ $(\Delta x)^2$]

If we now consider taking <u>continuum limit</u> where $a \to 0$, we will obtain a continuous infinity of degrees of freedom

$$L = T - V \xrightarrow{a\to 0} \int dx \left[\frac{1}{2}\frac{m}{a}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2} ka \left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{1}{a} v(\phi) \right] \quad *$$

where $\sum_i \to \int \frac{dx}{a}$

$\phi = \phi(x)$ $\quad q_i = \phi(x = x^i = ia)$; $q_{i+1} = \phi(x = (i+1)a)$

So $(q_{i+1}-q_i) = \phi((i+1)a) - \phi(ia) = \Delta\phi$

$\therefore$ $\lim a\to 0 \left(\frac{\Delta\phi}{a}\right) = \frac{d\phi}{dx}$ and we obtain *

Here it is assumed that $a \to 0$ limit $m \to 0$, $k \to \infty$ $v \to 0$ such that m/a, ka and v/a remain <u>finite</u>.

The net result of this limit is a <u>Lagrangian field Theory</u>

If we generalize to 3d and also impose Lorentz invariance on the resulting action

$$A = \int dt\, L \quad ; \quad L = \int d^3x\, \mathcal{L}$$

↑ Lagrangian density

$$\mathcal{L} = \mathcal{L}(\phi(\vec{x},t), \partial_\mu \phi)$$

E-L Equations $\Rightarrow$ $$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\partial^\mu = \left(\frac{1}{c} \partial/_{\partial t}, \vec{\nabla} \right) \quad ;$$

As an example consider

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) - \frac{k^2}{2} \phi^2$$

E-L eqs $\Rightarrow$ $$\boxed{\partial^\mu \partial_\mu \phi + k^2 \phi = 0}$$

Called Klein-Gordon equation.

Note that compared to e.g. point particle Lagrangian $\mathcal{L}$ has to be Lorentz invariant, since $\int dt \int d^3x$ is <u>invariant</u> under L.T. [check as exercise!]

So what might be the action / Lagrangian density of the Maxwell field?

Constraints 1) must be Lorentz scalar

2) must be gauge invariant.

2) $\Rightarrow$ Built from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

1) $\Rightarrow$ must be a function of $F_{\mu\nu} F^{\mu\nu}$.

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \quad ; \quad S = \int d^4x \, \mathcal{L}.$$

$$\delta S = \int d^4x \left(-\frac{2}{4\mu_0} F_{\mu\nu} \delta F^{\mu\nu} \right)$$

$$= \int d^4x \left(-\frac{1}{2\mu_0} F_{\mu\nu} \left(\partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu \right) \right)$$

$$\overset{I.P}{=} \int d^4x \left\{ + \frac{1}{2\mu_0} \partial^\mu F_{\mu\nu} \delta A^\nu \right\} \times 2$$

$\delta S = 0$ for all $\delta A^\nu \Rightarrow$

$$\boxed{\partial^\mu F_{\mu\nu} = 0} \Rightarrow \text{Source free Maxwell Eqn's}$$

[or Equivalently use E-L equations:- $\partial_\mu \left(\frac{\delta \mathcal{L}}{\partial_\mu A_\nu} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu}$]

How to include source term?

Easy: $\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - \underbrace{A_\nu J^\nu}$.

Euler-Lagrange equations $\Rightarrow$

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu}$$ — Exactly covariant Maxwell Equations with source.

What about gauge invariance when $J^\nu \neq 0$?

Although $\delta_\Lambda \mathcal{L} \neq 0$; $\delta_\Lambda \mathcal{L} = -\frac{1}{2\mu_0} F_{\mu\nu}\, \delta_\Lambda (F^{\mu\nu}) - (\delta_\Lambda A_\nu) J^\nu$

$\delta_\Lambda A_\nu = \partial_\nu (\delta\Lambda)$; $\delta\Lambda(x^\mu)$ – arb. scalar function.

$\Rightarrow \delta_\Lambda (F_{\mu\nu}) = 0$; $\delta_\Lambda \mathcal{L} = -\partial_\nu (\delta\Lambda) J^\nu$.

But $\delta_\Lambda S = \int d^4x\, \delta_\Lambda \mathcal{L} = \int d^4x \left(-\partial_\nu(\delta\Lambda) J^\nu\right)$

$$\overset{I.P}{=} \int d^4x\, \delta\Lambda \left(\partial_\nu J^\nu\right)$$

However using conservation of charge: $\boxed{\partial_\nu J^\nu = 0}$

$\Rightarrow$ S is gauge invariant!

Stress Tensor

We seek a covariant expression for the stress energy tensor of the electromagnetic field. Such a tensor should be conserved (with or without presence of sources J^μ) and it should have $T^{00} \sim$ energy density u of the E-M field, while $T^{0i} \sim$ Poynting vector S^i, as we have discussed in previous lectures.

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu .$$

– canonical Hamiltonian $H = \int d^3x \, \mathcal{H}$

where $\mathcal{H} = \frac{\delta \mathcal{L}}{\partial A_{\mu,0}} A_{\mu,0} - \mathcal{L}$ $\quad$ ["$H = p\dot{q} - L$"]

$$\Rightarrow \quad \mathcal{H} = \underbrace{\frac{1}{2} \epsilon_0 (\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B})}_{\text{recognize as } u} + \vec{\nabla}\cdot(\epsilon_0 \Phi \vec{E}) - \vec{J}\cdot\vec{A}$$

A 'guess' for canonical $T^\mu{}_\lambda$ is:-

$$T^\nu{}_\lambda = \frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} A_{\mu,\lambda} - \delta^\nu_\lambda \mathcal{L} = \frac{1}{\mu_0} F^{\mu\nu} A_{\mu,\lambda} - \delta^\nu_\lambda \mathcal{L} .$$

here $A_{\mu,\nu} \equiv \partial_\nu A_\mu$

$\Rightarrow \quad T^{00} = H$ as required.

But there are 'problems' with this definition of $T^{\nu}{}_{\lambda}$:-

1) It's not gauge invariant. Yet it should be as $T^{\nu}{}_{\lambda}$ encodes physical quantities like energy density, momentum density etc.. carried by the field.

2) $T^{\nu}{}_{\lambda}$ is not conserved in presence of sources.

Consider
$$\partial_\mu T^{\mu\nu} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - \eta^{\mu\nu} \mathcal{L} \right\}$$
$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \right) \partial^\nu A_\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial_\mu \partial^\nu A^\rho$$
$$- \partial^\nu \mathcal{L}.$$

Using equation of motion (with $J^\mu = 0$).

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \right) = \frac{\partial \mathcal{L}}{\partial A_\rho} \quad \Rightarrow$$

$$\partial_\mu T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial A_\rho} \partial^\nu A_\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu (\partial_\mu A_\rho) - \partial^\nu \mathcal{L}$$

$= 0$ since 1^{st} 2 terms $= \partial^\nu \mathcal{L}$ by chain rule.

$\therefore \quad \partial_\mu T^{\mu\nu} = 0$ but only if $J^\nu = 0$

Improved Stress Tensor

These various issues (and others we have not mentioned - see typed notes) are solved by taking the following definition for stress tensor:-

$$\Theta^{\mu\nu} = -\frac{1}{\mu_0}\left\{ F^{\rho\mu} F^{\lambda\nu} \eta_{\rho\lambda} - \frac{1}{4}\eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right\}.$$

Notice several properties:-

1) $\Theta^{\mu\nu} = \Theta^{\nu\mu}$ - symmetric tensor

2) traceless $\Theta^{\mu}{}_{\mu} \equiv \Theta^{\mu\nu}\eta_{\mu\nu} = -\frac{1}{\mu_0}\left(F^{\rho\mu} F_{\rho\mu} - \frac{1}{4}\overbrace{\eta^{\mu\nu}\eta_{\mu\nu}}^{=-1} \times F^{\rho\sigma} F_{\rho\sigma} \right)$
$= 0.$

3) It's obviously gauge invariant since $F^{\mu\nu}$ is.

4) $\Theta^{00} = \frac{1}{2}\epsilon_0(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B})$; $\Theta^{0i} = \frac{1}{c}S^i = \frac{1}{c\mu_0}(\vec{E}\times\vec{B})^i$
$= c\underbrace{g^i}_{\text{momentum density}}$

and $\Theta^{ij} = -\epsilon_0\left\{ E^iE^j + c^2B^iB^j - \frac{1}{2}\delta^{ij}(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}) \right\}$

$= -$ [Maxwell Stress tensor] (see earlier)

Conservation Laws

$$\partial_\mu \Theta^{\mu\nu} - J^\lambda F_\lambda{}^\nu = 0$$

(See HW8) . Proof makes use of Equations of motion

$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$ and the Maxwell field strength identity

$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0$.

Now call $-J^\lambda F_\lambda{}^\nu = f^\nu$.

Consider
$$f^{\nu=0} = J^\lambda F_\lambda{}^0 = \overset{0}{\overbrace{J^0 F_0{}^0}} + J^i F_i{}^0$$
$$= \frac{1}{c} \vec{E}\cdot\vec{J}$$

and
$$f^{\nu=i} = J^\lambda F_\lambda{}^i = J^0 F_0{}^i + J^j F_j{}^i$$
$$= \rho E^i + (\vec{J}\times\vec{B})^i$$

Recall that Lorentz force acting on point charge q:

$$\vec{F} = q(\vec{E} + \vec{V}\times\vec{B})$$

and rate of work done by $\vec{E}$ field acting on charge is

$$\frac{dW}{dt} = \vec{F}\cdot\vec{V} = \vec{E}q\cdot\vec{V}$$

So we see that 4-vector f^μ gives rate of change of <u>energy</u> and <u>momentum</u> of <u>source</u>:

$$\boxed{\int d^3x\, f^0 = \frac{1}{c}\int \vec{E}\cdot\vec{J}\, d^3x = \frac{1}{c}\frac{dE_{matter}}{dt}}$$

(if $\vec{J} = q\,\vec{V}\,\delta^{(3)}(\vec{x}-\vec{r}(t))$; $\frac{d\mathcal{E}_{matt}}{dt} = q\,\vec{E}\cdot\vec{V}$)

and $$\boxed{\int d^3x\, f^i = \int d^3x\,(\rho E^i + (\vec{J}\times\vec{B})^i)}$$

(so if $\rho = q\,\delta^{(3)}(\vec{x}-\vec{r}(t))$ and $\vec{J}$ is as above,

$$\int d^3x\, f^i = \int d^3x\,\left(q\,\delta^{(3)}(\vec{x}-\vec{r}(t))\,E^i + q\,\delta^{(3)}(\vec{x}-\vec{r}(t))\,(\vec{V}\times\vec{B})^i\right)$$
$$= q\left(E^i + (\vec{V}\times\vec{B})^i\right)$$)

Now the interpretation of conservation law

$\partial_\mu \Theta^{\mu\nu} + f^\nu = 0$ is clear:

$$\int (\partial_\mu \Theta^{\mu\nu} + f^\nu)\, d^3x = 0$$

$$= \int \left(\partial_0 \Theta^{0\mu} + \partial_i \Theta^{i\mu}\right) d^3x + \int d^3x\, f^\mu = 0$$

$$= \frac{d}{dt}P^\mu_{field} + \frac{d}{dt}P^\mu_{matter} + \text{surface term} = 0$$

where $\dfrac{dP^\mu_{matter}}{dt} = f^\mu$ (as we have shown above)

and $P^\mu_{field} = \int d^3x\, \theta^{0\mu} = \left(\dfrac{E_{field}}{c}, \vec{P}_{field}\right)$

$$E_{field} = \int d^3x \frac{\epsilon_0}{2}\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right)$$

$$\vec{P}_{field} = \frac{1}{c}\int d^3x\, \vec{S} \qquad \left(\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \text{ - Poynting vector}\right)$$

is clearly energy and momentum associated with the EM field. Thus $\int d^3x\left(\partial_\mu \theta^{\mu\nu} + f^\nu\right) = 0$ expresses conservation of total 4-momentum $P^\mu_{matter} + P^\mu_{field} = \text{constant}$.

In terms of local conservation equations:

$\boxed{\nu = 0}$ $$\frac{1}{c}\left(\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S}\right) = -\frac{\vec{E}\cdot\vec{J}}{c}$$

$\boxed{\nu = i}$ $$\frac{\partial g^i}{\partial t} = \left(T^{Maxwell}\right)^{ij}{}_{,j} - \left(\rho E + (\vec{J}\times\vec{B})^i\right)$$

$g^i = \frac{1}{c}S^i$ = momentum density of EM field