Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-966(4261) ELECTROMAGNETIC THEORY

Time Allowed: 2 hours 30 minutes

Date: 06th May, 2011

Time: 10.00 - 12.30

Instructions: **Answer ONLY THREE questions. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper. Course work comprises 10% of the final mark.**

A formula sheet containing mathematical results that may be of help in various questions is provided at the end of the examination paper.

Numeric calculators are not permitted in this examination. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr V Gili, Dr D Berman

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Question 1

a. Consider a slab of dielectric placed in an external static electric field. The electric dipole moment of the bound charges can be represented macroscopically by a polarization vector $\mathbf{P}$. What does $\mathbf{P}$ represent? How is $\mathbf{P}$ related to the density of the bound charges ρ_b. **[2]**

b. Now suppose you switch on a uniform and static magnetic field. What does the magnetization vector $\mathbf{M}$ represent? How is it related to the current of bound charges in the dielectric? **[2]**

c. Use the relation between $\mathbf{P}$ and ρ_b and the relation between $\mathbf{M}$ and $\mathbf{J}_b$ that you wrote in parts a. and b. respectively to show that in the case of stationary electric and magnetic fields the Gauss and Ampére laws read:

$$\nabla \cdot \mathbf{D}(\mathbf{x}) = \rho(\mathbf{x})_{free} \qquad \text{and} \qquad \nabla \times \mathbf{H} = \mathbf{J}(\mathbf{x})_{free},$$

where, for a linear and isotropic media, $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$. **[3]**

d. Describe how the configuration of charges and fields in the matter changes, with respect to the electrostatic and magnetostatic cases, when we place the dielectric in time varying electric and magnetic fields. Show that in this case the Maxwell equations in the matter and in the presence of sources are:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{D} = \rho_{free}, \qquad \nabla \times \mathbf{H} = \mathbf{J}_{free} + \frac{\partial \mathbf{D}}{\partial t}.$$

[5]

e. Consider a region of space V bounded by a closed surface S, and also let C be a closed contour in space with an open surface S' spanning the contour. Explaining the notation used, integrate over these region the Maxwell equations in the matter, and rewrite them as:

$$\int_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV \qquad \int_C \mathbf{H} \cdot d\mathbf{l} = \int_{S'} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}'$$

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \int_C \mathbf{E} \cdot d\mathbf{l} = -\int_{S'} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}'$$

[4]

f. Consider two regions, labelled by $i = 1, 2$, containing different linear media, which meet at an infinite two-dimensional boundary, with unit normal $\hat{\mathbf{n}}$ to the boundary. Let $\mathbf{E}_i$, $\mathbf{B}_i$, $\mathbf{D}_i$, $\mathbf{H}_i$ for $i = 1, 2$ label the electromagnetic fields in the two regions. Using a suitable small, shallow cylinder, straddling the boundary between the two regions, with surface charge density σ, derive the boundary conditions

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{\mathbf{n}} = \sigma, \qquad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \hat{\mathbf{n}} = 0$$

from two of the integral equations above.
Now considering a suitable small rectangle straddling the boundary, with current density $\mathbf{K}$ on the surface of the rectangle, derive the further boundary conditions

$$\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} \qquad \hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$$

[6]

Question 2

In the Lorentz gauge, the equation for the vector potential is:

$$\Box \mathbf{A}(\mathbf{x},t) = -\mu_0 \mathbf{J}(\mathbf{x},t). \tag{1}$$

a. Integrate the equation (1) with $\int_{-\infty}^{+\infty} e^{i\omega t} dt$ to obtain the Fourier transformed equation

$$\left(\nabla^2 + k^2\right) \mathbf{A}(\mathbf{x},\omega) = -\mu_0 \mathbf{J}(\mathbf{x},\omega)$$

What is the relation between k and ω? **[3]**

b. Suppose that there exists a Green function $G_k(\mathbf{x},\mathbf{x}')$ satisfying

$$\left(\nabla^2 + k^2\right) G_k(\mathbf{x},\mathbf{x}') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}').$$

Write an integral representation for $\mathbf{A}(\mathbf{x},\omega)$ in terms of the $G_k(\mathbf{x},\mathbf{x}')$ and of the current density $\mathbf{J}(\mathbf{x},\omega)$ and show that this satisfies (1). **[3]**

c. Explain why you can assume that $G_k(\mathbf{x},\mathbf{x}')$ is function only of the distance $r = |\mathbf{x}-\mathbf{x}'|$ and, using polar spherical coordinates and considering the cases $r \neq 0$ and $r \to 0$ separately, show that an expression for $G_k(r)$ is given by the following combination of the two particular solutions $G_k^{(\pm)}(r) = \frac{e^{\pm ikr}}{r}$:

$$G_k(r) = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r} \qquad \text{with} \quad A + B = 1$$

[7]

d. The time-dependent Green functions obtained from $G_k^{(\pm)}(r)$ are:

$$G^{(\pm)}(\mathbf{x},t;\mathbf{x}',t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'| - i\omega(t-t')}}{|\mathbf{x}-\mathbf{x}'|} d\omega.$$

Introduce a suitable definition of $\mathbf{A}(\mathbf{x},t)$ in terms of $G^{(\pm)}(\mathbf{x},t;\mathbf{x}',t')$ and use this expression to write the retarded and advanced vector potentials, and explain their physical significance. **[7]**

Question 3

a. In the derivation of scattering of sunlight from air molecules the equation relating the electric displacement field $\mathbf{D}(\mathbf{x})$ to the scatterer $\mathbf{s}(\mathbf{x})$

$$\left(\nabla^2 + k^2\right)\mathbf{D}(\mathbf{x}) = \mathbf{s}(\mathbf{x})$$

has solution

$$\mathbf{D}(\mathbf{x}) = \mathbf{D}_0(\mathbf{x}) - \frac{1}{4\pi}\int d^3y \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\mathbf{s}(\mathbf{y}) \qquad (2)$$

where $\mathbf{D}_0$ is a solution of the associated homogeneous equation and k is the wave number of a plane wave propagating along the direction $\mathbf{k} = k\hat{\mathbf{n}}$. Explain the derivation of the solution (2) using the retarded Green function $G_k^{(+)}(|\mathbf{x}-\mathbf{y}|) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$. Show that in the far zone, where $r >> d$, where d are the linear dimensions of the scatterer, you can expand $|\mathbf{x}-\mathbf{y}|$ in powers of $\frac{|\mathbf{y}|}{r}$ to get the simplified solution:

$$\mathbf{D}(\mathbf{x}) = \mathbf{D}_0(\mathbf{x}) - \frac{e^{ikr}}{4\pi r}\int d^3y e^{ik\hat{n}\cdot\mathbf{y}}\mathbf{s}(\mathbf{y}). \qquad (3)$$

[7]

b. Neglecting the magnetization effects, one can write $\mathbf{s}(\mathbf{x}) = -\nabla\times\nabla\times\mathbf{P}$, where $\mathbf{P}$ is the polarization vector. Explain how the Born approximation leads to

$$\mathbf{P}(\mathbf{x}) = \delta\epsilon_r(\mathbf{x})\, D_0\hat{\boldsymbol{\epsilon}}_0 e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}}$$

for an incident wave $\mathbf{D}_0 = D_0\hat{\boldsymbol{\epsilon}}_0 e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}}$, where $\delta\epsilon_r(\mathbf{x}) = \frac{\delta\epsilon(\mathbf{x})}{\epsilon_0}$ is the relative deviation of the electric permittivity from the empty space value ϵ_0. **[7]**

c. Deduce that in the Born approximation and in the far zone the scattered part of the displacement field in (3) takes the form

$$-\frac{D_0 e^{ikr}}{r}\frac{k^2}{4\pi}\int d^3y e^{i\mathbf{q}\cdot\mathbf{y}}\hat{\mathbf{n}}\times\hat{\mathbf{n}}\times\hat{\boldsymbol{\epsilon}}_0\,\delta\epsilon_r(\mathbf{x})$$

.

[6]

Question 4

Consider the Maxwell equations in a vacuum with sources:

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \tag{4}$$

a. Show that expressing the electric and magnetic field in term of a scalar and a vector potential ϕ and $\mathbf{A}$ as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

the two homogeneous Maxwell equations in (4) are automatically solved and that the two inhomogeneous ones reduce to:

$$\nabla^2\phi + \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0} \tag{5a}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0 \mathbf{J} \tag{5b}$$

[4]

b. Show that the electric and magnetic fields are unchanged if we perform the following gauge transformations on the potentials:

$$\phi \longrightarrow \phi - \frac{\partial \chi}{\partial t} \tag{6a}$$

$$\mathbf{A} \longrightarrow \mathbf{A} + \nabla\chi, \tag{6b}$$

where $\chi(\mathbf{x}, t)$ is any scalar function of $\mathbf{x}$ and t. **[4]**

c. Define the Lorentz gauge fixing condition, and rewrite the Maxwell equations (5) in this gauge. Use this result to explain why the Lorentz gauge is also called *radiation gauge.* **[4]**

d. Consider the Lorentz covariant notation and rewrite the Maxwell equations in (5) in term of the four-potential $A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right)$.
Rewrite the gauge transformations (6) in terms of A^μ.
How does the Lorentz gauge condition read in this case? **[4]**

e. The Maxwell equations in terms of the electromagnetic Field Strength are:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad \text{and} \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0.$$

Show that the potential representation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ automatically satisfies the second of these equations. **[4]**

Question 5
In the presence of sources the Lagrangian density for the electromagnetic field is:

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - j_\alpha A^\alpha$$

a. Show that the Euler-Lagrange equations:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

are $\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$. **[4]**

b. The canonical stress tensor expressed in terms of the Lagrangian density $\mathcal{L}$ is:

$$T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial \partial_\nu A_\alpha} \partial_\mu A_\alpha - \delta^\nu_\mu \mathcal{L}$$

Use this definition to prove that $\partial_\nu T^\nu_\mu = 0$, then derive an expression for T^ν_μ in terms of the tensors F and A. **[4]**

c. Show that in the absence of sources the canonical stress tensor $T^{\mu\nu}$ differs from the symmetric stress tensor

$$\Theta^{\mu\nu} = -\frac{1}{\mu_0} \left[F^{\alpha\mu} F_\alpha{}^\nu - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

by a total derivative. **[4]**

d. Show that

$$\Theta^{00} = \frac{1}{2}\epsilon_0 (\mathbf{E}^2 + c^2 \mathbf{B}^2) \qquad \Theta^{0i} = \frac{1}{c\mu_0} (\mathbf{E} \times \mathbf{B})^i$$

and give the physical significance of these quantities. **[4]**

e. Show that in the presence of sources $\partial_\mu \Theta^{\mu\nu} = j^\mu F_\mu{}^\nu$. Then show that for $\nu = 0$ this equation becomes the Poynting equation:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{P}} = -\mathbf{E} \cdot \mathbf{J}, \qquad (7)$$

where $\mathcal{E}$ is the energy density of the electromagnetic field and $\boldsymbol{\mathcal{P}}$ is the Poynting vector. Then explain the physical significance of (7). **[4]**

FORMULA SHEET

$$
\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad & = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \\
\nabla \cdot (\psi \mathbf{a}) \quad & = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}, \\
\nabla \times (\psi \mathbf{a}) \quad & = (\nabla \psi) \times \mathbf{a} + \psi (\nabla \times \mathbf{a}), \\
\nabla \times (\nabla \times \mathbf{a}) \quad & = \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}, \\
\nabla \big(\psi(r) \big) \quad & = \mathbf{n} \psi'(r).
\end{aligned}
$$

Metric:

$$
\eta_{\alpha\beta} = \begin{cases} +1 & \text{if } \alpha = \beta = 0 \\ -1 & \text{if } \alpha = \beta = 1, 2, 3 \\ 0 & \text{if } \alpha \neq \beta \end{cases}
$$

$$
\| F^{\alpha\beta} \| = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}.
$$

In spherical coordinates (r, θ, ϕ), with corresponding unit coordinate vectors $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi})$, for a vector field $\mathbf{A}$ with components (A_r, A_θ, A_ϕ),

$$
\begin{aligned}
\nabla \times \mathbf{A} = \quad & \hat{\mathbf{r}} \, \tfrac{1}{r \sin\theta} \left(\tfrac{\partial}{\partial \theta} (A_\phi \sin\theta) - \tfrac{\partial A_\theta}{\partial \phi} \right) + \hat{\theta} \left(\tfrac{1}{r \sin\theta} \tfrac{\partial A_r}{\partial \phi} - \tfrac{1}{r} \tfrac{\partial}{\partial r} (r A_\phi) \right) \\
& + \hat{\phi} \tfrac{1}{r} \left(\tfrac{\partial}{\partial r} (r A_\theta) - \tfrac{\partial A_r}{\partial \theta} \right)
\end{aligned}
$$

and for a scalar field $G(r, \theta, \phi)$

$$
\nabla^2 G = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rG) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial G}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 G}{\partial \phi^2}.
$$