November 5, 2012

PHY 217: Vibrations and Waves

Course handout material

1 Harmonic Motion, SHO

A simple harmonic oscillation is described by:

$$x = A\cos(\omega t + \phi) \tag{1}$$

where t is time, A is the amplitude, ω [rad/s] is the angular frequency, $T = 2\pi/\omega$ [s] is the period, $f = 1/T$ [Hz] is the frequency. Harmonic motion can be obtained from the projection of circular uniform motion onto any axis.

1.1 Derivation of SHO equation: mass on spring

Consider a mass attached to an horizontal spring, on a frictionless plane. The equation:

$$\vec{F} = m\vec{a} \tag{2}$$

in one dimension and considering Hooke's law becomes:

$$F = -kx \tag{3}$$

$$m\ddot{x} = -kx \tag{4}$$

$$\ddot{x} + \frac{k}{m} = 0 \tag{5}$$

Note that the dimension of k is $[k] = [kg\frac{m}{s^2}m^{-1}] = \frac{kg}{s^2}$, therefore $[k/m] = [1/s^2]$ which is a frequency f squared.
For a mass attached vertically to a spring (therefore under gravitational force), the equilibrium point shifts downwards, but the equation (SHO) is the same as the above.

1.2 Solutions and general solution

A function of form: $x(t) = A\cos(\omega_0 t)$ with $A \in \mathbb{R}$ is a solution to the SHO equation, provided that $\omega_0 = \sqrt{k/m}$ (substitute into the equation to verify this), but not a general solution. A general solution is independent of initial conditions, and to obtain the general solution we can modify the previous by adding a constant that adjusts the phase:

$$x(t) = A\cos(\omega_0 t + \phi) \tag{6}$$

with A, ϕ constants. The condition on ω_0 is the same as above. The constants can be specified by giving initial conditions on position and velocity, x_0 and v_0, at a specified time, e.g. $t = 0$.

$$\begin{cases} x_0 = A\cos\phi \\ v_0 = -\omega_0 A\sin\phi \end{cases} \tag{7}$$

from which we obtain:

$$\phi = \mathrm{atan}\left(\frac{-v_0}{x_0\omega_0}\right) \tag{8}$$

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}} \tag{9}$$

$$\tag{10}$$

1.3 Complex representation

The SHO can be derived from the projection onto a diameter of a uniform rotating motion, and we can also describe the uniform rotating motion using complex numbers (see fig 1). The equation is: $\ddot{z} + \omega_0^2 z = 0$ with $z \in \mathbb{C}$. The solution that we are interested in can be otained as the $Re[z] = x$. As a solution to the SHO equation in the complex plane, we try:

$$z(t) = Ae^{i(pt+\phi)} \tag{11}$$

where $A,\ \phi \in \mathbb{R}$ and $p \in \mathbb{C}$ in general. This is a solution if $p = \omega_0$, and we can obtain the solution in the real axis as:

$$Re[z] = Re\left[Ae^{i(\omega_0 t+\phi)}\right] = A\cos(\omega_0 t + \phi) \tag{12}$$

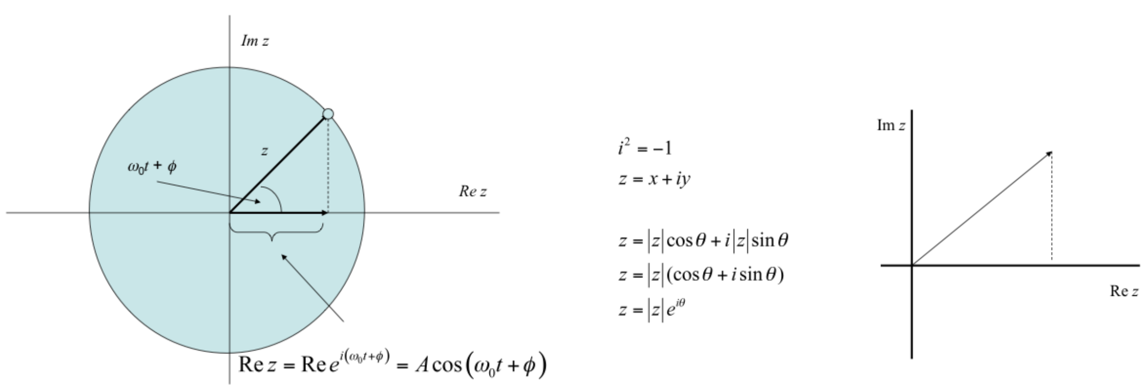


Figure 1: Complex representation.

1.4 Other SHO Systems: pendulum

Consider a simple pendulum of mass m, length L oscillating with small angles, we can project the motion onto the x axis, the equation is:

$$m\ddot{x} = F_x = -T\sin\theta \simeq -mg\theta \tag{13}$$

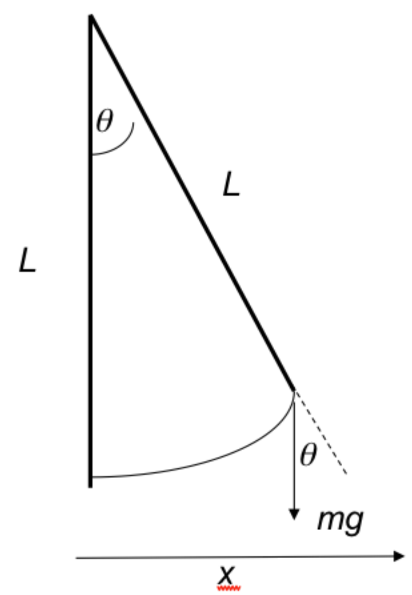


Figure 2: Pendulum.

hence:

$$\ddot{x} + \frac{g}{L}x = 0 \tag{14}$$

which is the same form of a SHO with $\omega_0 = \sqrt{g/L}$, the period is therefore $T = 2\pi\sqrt{L/g}$ independent on the mass, and independent on the initial conditions.

1.5 Bobbing cork

Consider a liquid of density ρ, and a cork of cross section area A. The restoring force is $F = -\rho Axg$ and the equation of motion:

$$m\ddot{x} = -\rho Agx \tag{15}$$

$$\ddot{x} + \frac{\rho Ag}{m}x = 0 \tag{16}$$

$$\tag{17}$$

which is a SHO equation with $\omega_0^2 = (\rho Ag)/m$. If the mass m is expressed in terms of the draft: $m = \rho Ah$ then $\omega_0^2 = g/h$.

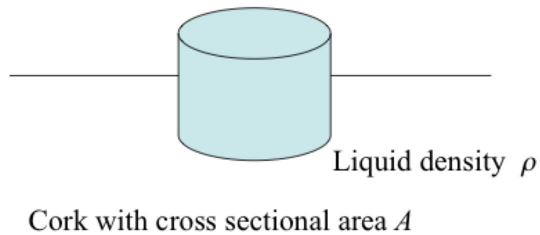


Figure 3: Bobbing cork.

1.6 LC circuit

Consider charging up a capacitor C and then discharging through an inductor, L:

$$V_L = -L\dot{I} \qquad V_c = \frac{Q}{C} \qquad I = \frac{dQ}{dt} = \dot{Q} \tag{18}$$

$$-L\dot{I} = \frac{Q}{C} \Rightarrow \dot{I} = \frac{1}{LC}Q \tag{19}$$

$$\ddot{I} = -\frac{1}{LC}I \tag{20}$$

$$\ddot{I} + \frac{1}{LC}I = 0 \tag{21}$$

which is SHO, the current oscillates through the capacitor, with $\omega_0^2 = 1/(LC)$.

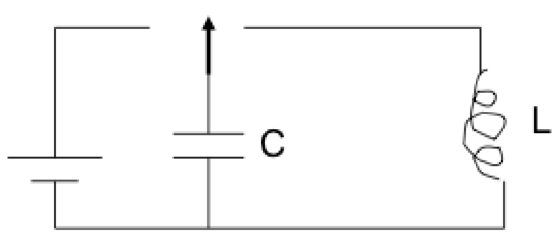


Figure 4: LC circuit.

1.7 Physical pendulum

The equation for a physical pendulum can be derived from the rotation equivalent of the Newton's II Law:

$$\vec{\tau} = I\vec{\alpha} \tag{22}$$

with $\vec{\tau}$ the torque, I the moment of inertia and $|\vec{\alpha}| = \ddot{\theta}$ the angular acceleration. The torque of the force at the center of mass (O), relative to the point around which the pendulum rotates (P), is $\vec{\tau} = \vec{r} \times \vec{F} = -bmg\sin\theta$, b is the distance between O and P, hence:

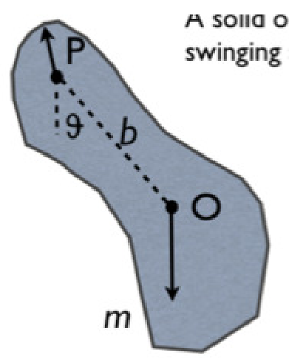


Figure 5: Physical pendulum.

$$-bmg\sin\theta = I_p\ddot{\theta} \tag{23}$$

and for small angles:

$$\ddot{\theta} + \frac{bmg}{I_p}\theta = 0 \tag{24}$$

which is SHO with $\omega_0 = \sqrt{bmg/I_p}$. One can easily calculate the period of oscillations of a hoop, which has moment of inertia $I_p = 2mR^2$.

1.8 Energy in SHO: mass on a spring

Integrating both sides of the SHO equation for a mass on spring, $m\ddot{x} = -kx$ one obtaines:

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E \tag{25}$$

The first term is the kinetic energy, the second the potential energy and E is a constant, the energy of the oscillating system. Since x is a function of time t, the total energy can be written as:

$$E_{TOT} = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) \tag{26}$$

$$= \frac{1}{2}A^2k\left[\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)\right] = \frac{A^2k}{2} \tag{27}$$

$$\tag{28}$$

One can easily calculate the maximum speed ($E_p = 0$) and the maximum elongation ($E_k = 0$).

2 Damped Harmonic Motion

Consider a mass attached to a spring moving horizontally on a plane, but with a frictional force:

$$F = -r\dot{x} \tag{29}$$

so the total force acting on the mass is: $F = -kx - r\dot{x}$ and writing Newton's II law:

$$m\ddot{x} = -kx - r\dot{x} \;\Rightarrow\; \ddot{x} + \frac{k}{m}x + \frac{r}{m}\dot{x} = 0 \;\Rightarrow\; \ddot{x} + \omega_0^2 x + \gamma\dot{x} = 0 \tag{30}$$

where $k/m = \omega_0^2$ for analogy with the SHO and $r/m = \gamma$ $[\gamma] = s^{-1}$.

2.1 Solutions of the DHO equation: underdamping

It is easier to work with the complex representation (this is because instead of doing algebra with trigonometric functions we can use exponentials), write the DHO equation as:

$$\ddot{z} + \omega_0^2 z + \gamma\dot{z} \tag{31}$$

with $z \in \mathbb{C}$. Try the solution: $z(t) = A_0 e^{i(pt+\phi)}$ with A_0, $\phi \in \mathbb{R}$ and $p \in \mathbb{C}$, substituting $\dot{z}$ and $\ddot{z}$ into the equation we have:

$$-A_0 p^2 e^{i(pt+\phi)} + \omega_0^2 A_0 e^{i(pt+\phi)} + \gamma i A_0 p e^{i(pt+\phi)} = 0 \tag{32}$$

$$e^{i(pt+\phi)}\left[-A_0 p^2 + \omega_0^2 A_0 + \gamma i A_0 p\right] = 0 \tag{33}$$

$$A_0 e^{i(pt+\phi)}\left[-p^2 + \omega_0^2 + \gamma i p\right] = 0 \tag{34}$$

which implies:

$$p = \left(\frac{i\gamma}{2}\right) \pm \sqrt{-\left(\frac{\gamma}{2}\right)^2 + \omega_0^2} = i\frac{\gamma}{2} \pm \omega_d \tag{35}$$

substituting into $z(t)$:

$$z(t) = A_0 e^{[-\gamma/2 \pm i\omega_d]t + i\phi} = A_0 e^{-\frac{\gamma}{2}t} e^{i(\pm\omega_d t + \phi)} \tag{36}$$

and:

$$x(t) = Re[z(t)] = A_0 e^{-(\gamma/2)t}\cos(\omega_d t + \phi) \;\; or \;\; x(t) = A_0 e^{-t/\tau}\cos(\omega_d t + \phi) \tag{37}$$

with $\omega_d^2 = \omega_0^2 - \frac{\gamma^2}{4}$. It is also common to introduce the quality factor $Q = \omega_0/\gamma$ and write: $\omega_d^2 = \omega_0^2\left(1 - \frac{1}{4Q^2}\right)$.

2.2 Critical and over-damping

A dampened system has been described with:

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{38}$$

if $\gamma^2/4 > \omega_0^2$ then $\omega^2 < 0$ does not make sense. For such γ we need to go back to the original differential equation to find a new solution. We had (see the derivation done during the lecture or French):

$$n^2 = \omega_0^2 - \frac{\gamma^2}{4} \tag{39}$$

we can also write:

$$n = \pm i \left(-\omega_0^2 + \frac{\gamma^2}{4} \right)^{1/2} = \pm i\beta \tag{40}$$

substitute back into the equation we had ealier: $z(t) = A_0 e^{i(pt+\phi)}$ and $p = n + is$, so:

$$z(t) = A_0 e^{i([\pm i\beta + i\frac{\gamma}{2}]t+\phi)} = A_0 e^{\mp\beta t - \frac{\gamma}{2}t} e^{i\phi} \tag{41}$$

$$x(t) = A_1 e^{(-\frac{\gamma}{2}+\beta)t} + B_1 e^{(-\frac{\gamma}{2}-\beta)t} \qquad \beta = \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{42}$$

There are two adjustable constants describing all possible initial states, and nothing is oscillating anymore, there are two decay times, one for each term.
A special case is when $\omega_0 = \gamma/2$ (*critical damping*), the solution would seem to be: $n = 0$, $p = is = i\gamma/2$

$$z(t) = A_0 e^{i(i\frac{\gamma}{2}t+\phi)} = A_0 e^{-\gamma/2t} e^{i\phi} \tag{43}$$

but this is not the most general solution (it only has one adjustable constant). The most general solution solution is (substitute into the DHO equation to convince yourself this is indeed a solution)

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t} \tag{44}$$

Note that here there are two constants as expected. The critical damping is the quickest return to equilibrium without any oscillation.

2.3 RLC circuit

Consider an RLC circuit in series with a battery; a switch is closed and the capacitor starts charging up:

$$I = \frac{dQ}{dt} \qquad V_c = \frac{Q}{C} \qquad V_R = RI \qquad V_L = L\dot{I} \tag{45}$$

$$RI + L\dot{I} + \frac{Q}{C} = V_0 \Rightarrow R\dot{Q} + L\ddot{Q} + \frac{Q}{C} = V_0 \tag{46}$$

$$\ddot{Q} + \frac{Q}{LC} + \frac{R}{L}\dot{Q} = \frac{V_0}{L} \tag{47}$$

call $R/L = \gamma$ and $1/(LC) = \omega_0^2$:

$$\ddot{Q} + \gamma\dot{Q} + \omega_0^2 Q = \frac{V_0}{L}. \tag{48}$$

For $V_0 = 0$ the solution is a discharging capacitor:

$$Q(t) = Q_0 e^{-\frac{\gamma}{2}t}\cos(\omega t + \alpha) \tag{49}$$

with $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$. For $V_0 \neq 0$, the system will end up with a fully charged capacitor, and the solution:

$$Q = Q_0 e^{-\frac{\gamma}{2}t}\cos(\omega t + \alpha) + Q_{max} \tag{50}$$

with $Q_{max} = V_0 C$ should be solved for Q_0 and α. Possible initial conditions for $t = 0$ are $Q = 0$ (no charge in the capacitor) and $\dot{Q} = I = 0$ (no current flowing).

2.4 Energy in Damped Harmonic Motion

The energy for SHO has been written as:

$$E = \frac{1}{2} m\omega_0^2 A^2 \tag{51}$$

this now is modified by $A(t) = A_0 e^{-t/\tau}$ hence:

$$E = \frac{1}{2} m\omega_0^2 A_0^2 e^{-2t/\tau} = E_0 e^{-\gamma t} \tag{52}$$

with $\tau = 2/\gamma$. The quantity $\tau_E = 1/\gamma$ is the relaxation time for the energy. The quantity:

$$Q = \frac{1/\gamma}{1/\omega_0} = \frac{\omega_0}{\gamma} \tag{53}$$

is called *quality* of the oscillator. We can also find that the rate of energy loss equals the rate of work done against the frictional force, infact:

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 \Rightarrow \tag{54}$$

$$\frac{dE}{dt} = \frac{1}{2} m 2\ddot{x}\dot{x} + \frac{1}{2} kx\dot{x} = m\dot{x}\left(\ddot{x} + \frac{k}{m}x\right) \tag{55}$$

since for DHO: $\ddot{x} + \frac{r}{m}\dot{x} + \frac{k}{m}x = 0$ then $\ddot{x} + \frac{k}{m}x = -\frac{r}{m}\dot{x}$ and $F_{fric} = -r\dot{x}$ then:

$$\frac{dE}{dt} = F_{fric}\dot{x} \tag{56}$$

3 Forced Harmonic Oscillator

3.1 Mass on spring system

Consider a spring-mass system with spring constant k and mass m placed on an horizontal plane. There is friction, and the frictional force is $F_{fric} = -r\dot{x}$. Let us also impose a force to the mass: $F = F_0 \cos\omega t$. The equation of motion becomes:

$$m\ddot{x} = -kx - r\dot{x} + F_0 \cos\omega t \Rightarrow \ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t. \tag{57}$$

This is the equation that we must now solve. Use the method of the complex plane, the equation is:

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\omega t}. \tag{58}$$

and use as solution the trial function: $z(t) = Ae^{i(\omega t - \alpha)}$. Plugging $\ddot{z}$ and $\dot{z}$ in the equation gives:

$$-\omega^2 Ae^{i(\omega t-\alpha)} + \gamma i\omega e^{i(\omega t-\alpha)} + \omega_0^2 Ae^{i(\omega t-\alpha)} = \frac{F_0}{m} e^{i(\omega t-\alpha)} \tag{59}$$

$$\Rightarrow (-\omega^2 + i\gamma\omega + \omega_0^2)A = \frac{F_0}{m}(\cos\alpha + i\sin\alpha) \tag{60}$$

these are two equations for the Real and Imaginary parts:

$$(-\omega^2 + \omega_0^2)A = \frac{F_0}{m}\cos\alpha \tag{61}$$

$$\gamma\omega A = \frac{F_0}{m}\sin\alpha \tag{62}$$

square and add the equations to derive A and take the ratio to derive α:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}; \qquad \tan\alpha = \frac{\gamma\omega}{\omega_0^2 - \omega^2} \tag{63}$$

Therefore the trial function is a solution to the FHO equation in the complex plane, and these are the constants. The solution is the real part:

$$x(t) = A\cos(\omega t - \alpha) \tag{64}$$

Note that A and α are completely determined by the system (this is the *steady state* solution), there are no adjustable constants depending on the initial conditions as in SHO and DHO. This is due to the fact that the steady state solution has no memory of the initial conditions.

Note that $A \propto F_0$, $A \propto 1/m$ and $A \propto 1/\gamma$. Moreover, for $\omega_0 = 0$, $A = F_0/k$ and $\alpha = 0$. At the *resonance frequency* ($\omega = \omega_0$), the amplitude $A = QF_0/k$ (Q times

higher than the 0-frequency amplitude) and $\alpha = \pi/2$; and for the limit $\omega \to \infty$: $A \to \infty$ and $\alpha \to \pi$.
This curve $A(\omega)$ has actually a maximum at an angular frequency lower than the resonance frequency:

$$\omega_{max} = \sqrt{\left(\omega_0^2 - \frac{\gamma^2}{2}\right)} = \omega_0 \left(1 - \frac{1}{2Q^2}\right)^{1/2} \tag{65}$$

and the max is;

$$A_{max} = \frac{F_0}{k} \frac{Q}{\left(1 - \frac{1}{4Q^2}\right)^{1/2}} \tag{66}$$

However, already for modest Q values, $\omega_{max} \simeq \omega_0$ and $A_{max} \simeq QF_0/k$. For a complete discussion of the resonance mechanism please refer to your lecture notes, or to French.

3.2 Pendulum with forced oscillations

Consider a pendulum of mass m and length l, with frictional force $F_{fric} = -r\dot{x}$, subject to a shift horizontally of the top of the string: $s = s_0\cos(\omega t)$. Newtown's 2nd law for this system is:

$$m\ddot{x} = -r\dot{x} - T\sin\theta \tag{67}$$

For small oscillations $T \simeq mg$:

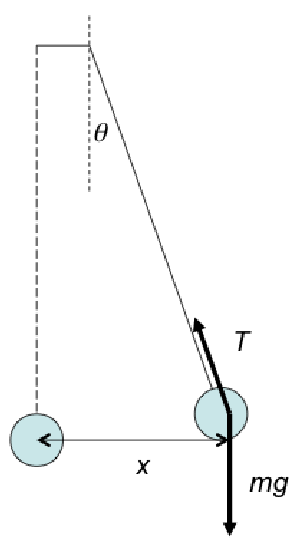


Figure 6: Pendulum with forced oscillations.

$$m\ddot{x} = -r\dot{x} - mg\sin\theta = -r\dot{x} - mg\frac{(x-s)}{l} \tag{68}$$

$$\Rightarrow m\ddot{x} = -r\dot{x} - mg\frac{x}{l} + \frac{mgs}{l} \tag{69}$$

$$\Rightarrow \ddot{x} + \gamma\dot{x} + \omega_0^2 x = \omega_0^2 s_0 \cos\omega t \tag{70}$$

where I have used: $\gamma = r/m$, $\omega_0^2 = g/l$. This is the same as the equation derived for the forced harmonic oscillator in the case of the mass attached to a spring, with the difference that $F_0/m \leftrightarrow \omega_0^2 s_0$. The solution is therefore the same, and the expressions for A and $\tan\alpha$ are the same (in A the numerator has $\omega_0^2 s_0$). You can easily use a pendulum and demonstrate the behaviour of A and α while you drive it at different frequencies ω.

3.3 Power absorbed by a driven oscillator

A driven oscillator subject to frictional forces absorbs energy. The work is $dW = \vec{F}\cdot\vec{x}$, and the power absorbed:

$$P = \frac{dW}{dt} = \vec{F}\cdot\vec{v} \tag{71}$$

and if we are considering the force and the motion along the same axis, this is simply $P = Fv$, where the force is the driver: $F = F_0\cos(\omega t)$. Therefore in steady state ($x = A\cos(\omega t - \alpha)$):

$$P(t) = Fv = -F_0 v\omega A\cos(\omega t)[\sin(\omega t - \alpha)] = \tag{72}$$

$$-F_0 v\omega A\cos(\omega t)[\sin(\omega t)\cos\alpha - \cos(\omega t)\sin\alpha] \tag{73}$$

However most times we are not interested in the instantaneous absorbed power, but the average power (averaged e.g. over a cycle):

$$\bar{p} = \frac{1}{t}\int_0^T P(t)dt \tag{74}$$

Since the first term in the square bracket is zero when integrated over a cycle, average power is:

$$\bar{P} = \frac{F_0}{2}\omega A\sin\alpha, \tag{75}$$

where α and A are known from the solution to the FHO:

$$\sin\alpha = \frac{\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \tag{76}$$

(derive this from $\tan\alpha$ using Pythagora's theorem). Hence:

$$\bar{P} = \frac{F_0\omega^2\gamma}{2m}\frac{1}{\left[\left(\omega_0^2-\omega^2\right)^2+\omega^2\gamma^2\right]} = \frac{F_0^2\gamma}{2m\left[\left(\frac{\omega_0^2}{\omega}-\omega\right)^2+\gamma^2\right]} \tag{77}$$

Notice that this curve has a maximum for $\omega = \omega_0$:

$$\bar{P}_{max} = \frac{F_0^2}{2m\gamma} = \frac{QF_0^2}{2m\omega_0} \tag{78}$$

and $\bar{P} \to 0$ in the limits for $\omega \to \infty,\ \omega \to 0$. The width at half max is approximately γ. Refer to your notes for a discussion on *resonance absorption*.

3.4 RLC circuit with drive

Suppose an RLC circuit in series is fed by an alternating voltage: $V_0\cos(\omega t)$, the equation for the circuit is:

$$RI + L\frac{dI}{dt} + \frac{Q}{C} = V_0\cos(\omega t) \Rightarrow L\ddot{I} + R\dot{I} + \frac{I}{C} = -V_0\omega\sin(\omega t) \tag{79}$$

$$\Rightarrow \ddot{I} + \gamma\dot{I} + \omega_0^2 I = -\frac{V_0\omega}{L}\sin(\omega t) \tag{80}$$

which looks very much like a FHO equation. The solution is: $I = I_0\sin(\omega t - \delta)$, with $I_0 \to 0$ for $\omega \to 0$ and $\omega \to \infty$, and maximum for $\omega = \omega_0$ ($I_0 = V_0/R$).

3.5 Superposition of oscillations: Beats

Before showing the full FHO solution, we need to consider the case of adding to oscillations. Let $x_1 = A\cos(\omega_1 t)$ and $x_2 = A\cos(\omega_2 t)$ be the solution to two SHO's, with the same amplitude A, and different frequency $\omega_{1,2}$. The superposition of the two is:

$$x(t) = x_1(t) + x_2(t) = A\cos(\omega_1 t) + A\cos(\omega_2 t) = \tag{81}$$

$$2A\cos\left[\frac{\omega_1+\omega_2}{2}t\right]\cos\left[\frac{\omega_1-\omega_2}{2}t\right] \tag{82}$$

when the frequencies $\omega_{1,2}$ are close to each other, the first cosine term describes an oscillation with frequency equal to the average of the two, whereas the second term describes a slow oscillation which can be interpreted as an amplitude modulation. The resulting motion is a fast oscillation modulated in amplitude, and the phenomenon is called *beats*. The period of the beat is defined as the time interval between two consecutive points in which the summed oscillations are in-phase, therefore the angular frequency of the beat is $\Delta\omega = \omega_1 - \omega_2$ (not devided by 2 as a first glance at the above equation might suggest).

3.6 Forced Harmonic Oscillations in transient state

The solution for a FHO in transient state can be written as the sum of the DHO solution and the steady state solution of the FHO:

$$x(t) = \hat{A}e^{-\frac{\gamma}{2}t}\cos(\omega' t + \alpha) + A\cos(\omega t - \delta) \tag{83}$$

where A, δ are completely determined by the system (FHO, steady state), and $\hat{A}, \alpha$ are the adjustable contants that depend on the initial condition. Infact, the first part gives 0 when plugged into the FHO equation, and so if the steady state solution is a solution to the FHO equation, so is this one. The sum of the two will describe the transient state, while the steady state solution dominates for $t >> 2/\gamma$. Figure 7 shows some examples.

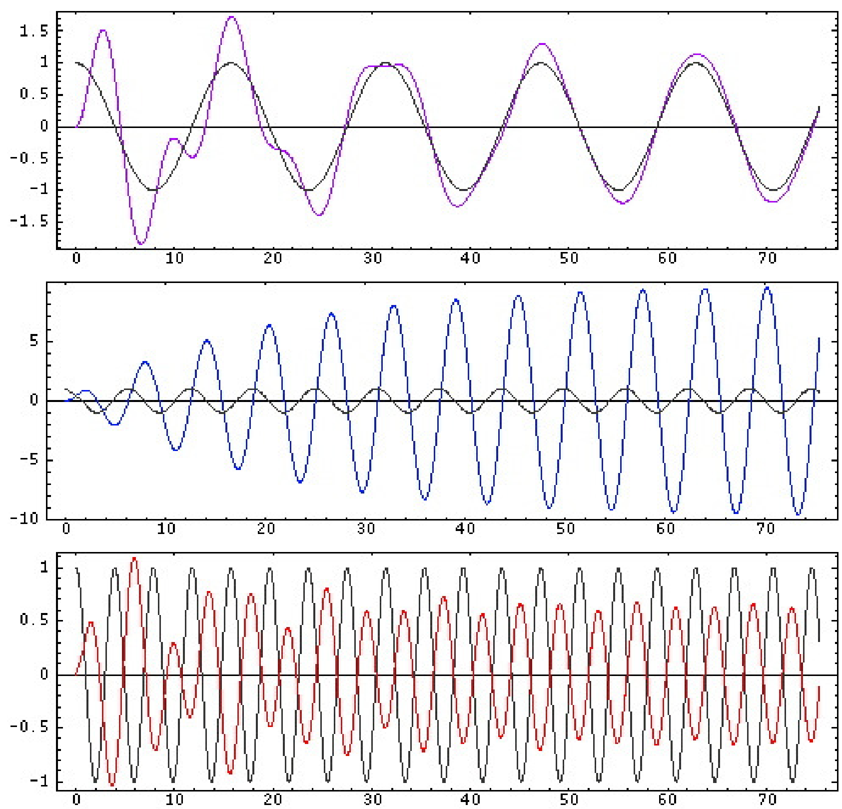


Figure 7: Graphs of the driving force (black) applied to a DHO (coloured). The DHO natural frequency is the same on all graphs while the force is applied with 3 different frequencies in the 3 graphs. Notice the transient and steady state regime and estimate where the force is above, at- or below resonance?

4 Coupled oscillators

4.1 Symmetry derivation

Consider two pendulums of same mass m and length l, connected by a spring of constant k (see Fig. 8. It is possible to predict the position and velocity of each at any given time and what any initial condition (there are 4 choices, 2 initial positions and 2 initial velocities). The general motion can be described as the linear superposition of two *normal modes*. A normal mode is a solution where both masses oscillate with the same frequency and are either *in-phase* or *180° out-of-phase*. So if I call ω_- the lowest frequency solution and ω_+ the higher frequency solution (there are 2 masses coupled and therefore there will be 2 normal modes), the position of the masses 1 and 2 at time t is:

$$x_1(t) = \chi_{0-}\cos(\omega_- t + \phi_-) + \chi_{0+}\cos(\omega_+ t + \phi_+) \tag{84}$$
$$x_2(t) = \chi_{0-}\cos(\omega_- t + \phi_-) - \chi_{0+}\cos(\omega_+ t + \phi_+) \tag{85}$$
$$\tag{86}$$

where $\chi_{0-,+}$ and $\phi_{-,+}$ are adjustable constants that depend on the initial conditions. These equation can be derived from symmetry considerations about the two normal modes of this system. In the lowest frequency mode, the amplitude of oscillation is the same, and in the higher frequency mode, the amplitude is the same and the sign takes care of the π phase difference between the two masses.

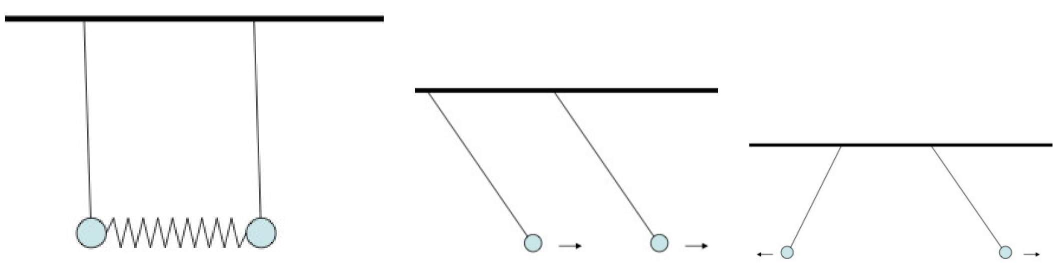


Figure 8: Coupled pundulum, and its two normal modes.

It is easy to calculate the angular frequency of the lowest frequency mode, as the spring never gets stretched nor compressed, so $\omega_- = \sqrt{g/\ell}$. For the higher frequency mode, a consideration of the forces acting on one of the masses (see Fig. 9) leads to:

$$m\ddot{x} = -2kx - mg\frac{x}{\ell} \Rightarrow \ddot{x} + 2\omega_s^2 x + \omega_0^2 x = 0 \tag{87}$$

with $\omega_s^2 = k/m$ and $\omega_0^2 = g/\ell$, from which it is simple to see that $\omega_+ = \sqrt{2\omega_s^2 + \omega_0^2}$. Note that the ratios of amplitudes is ± 1 for the two normal modes.

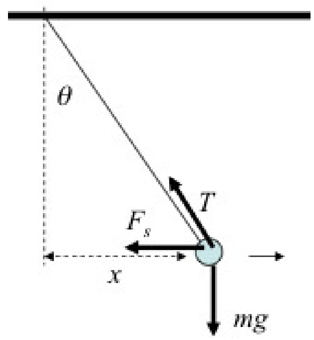


Figure 9: Forces acting on one of a coupled pendulum.

4.2 Beats in coupled pendulums

Suppose to choose the following initial conditions:

$$x_1 = C \tag{88}$$
$$\dot{x}_1 = 0 \tag{89}$$
$$x_2 = 0 \tag{90}$$
$$\dot{x}_2 = 0 \tag{91}$$
$$\tag{92}$$

meaning one of the masses is pulled and released from rest. By substituting into the general solutions, will find $\phi_- = 0$ and $\phi_+ = 0$ and:

$$x_1(t = 0) = C = \chi_{0-} + \chi_{0+} \tag{93}$$
$$x_2(t = 0) = 0 = \chi_{0-} - \chi_{0+} \Rightarrow \chi_{0+} = \chi_{0-} = \frac{C}{2} \tag{94}$$

hence:

$$x_1(t) = \frac{C}{2}\cos(\omega_- t) + \frac{C}{2}\cos(\omega_+ t) = C\cos\left(\frac{\omega_- + \omega_+}{2}t\right)\cos\left(\frac{\omega_- - \omega_+}{2}t\right) \tag{95}$$
$$x_1(t) = \frac{C}{2}\cos(\omega_- t) - \frac{C}{2}\cos(\omega_+ t) = C\sin\left(\frac{\omega_- + \omega_+}{2}t\right)\sin\left(\frac{\omega_- - \omega_+}{2}t\right) \tag{96}$$
$$\tag{97}$$

and if ω_- and ω_+ are not too far apart, this motion describes beats whereby the first mass slowly transfers energy to the second and then viceversa.

4.3 General prescription to derive normal modes

It is not always possible to identify the normal modes of a system solely based on symmetry; the following is however a prescription for calculating normal modes, by following these three steps:

- Give each mass a displacement from equilibrium
- Write Newton's IInd law for each mass
- Put the conditions for normal modes (all masses oscillating with the same frequency and either in-phase or 180° out-of-phase).

We now apply this prescription and derive again the normal modes for a coupled pendulum. Figure 10 shows step one. The differential equations are:

$$m\ddot{x}_1 = -mg\frac{x_1}{\ell} + k(x_2 - x_1) \tag{98}$$
$$m\ddot{x}_2 = -mg\frac{x_2}{\ell} - k(x_2 - x_1) \tag{99}$$
$$\tag{100}$$

rearranging:

$$\ddot{x}_1 + (\omega_0^2 + \omega_s^2)x_1 - \omega_s^2 x_2 = 0 \tag{101}$$
$$\ddot{x}_2 + (\omega_0^2 + \omega_s^2)x_2 - \omega_s^2 x_1 = 0 \tag{102}$$
$$\tag{103}$$

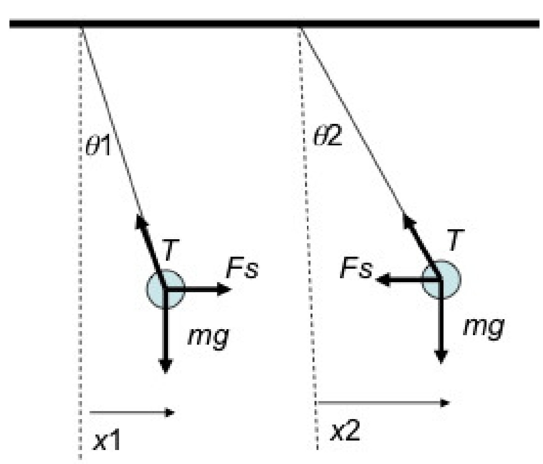


Figure 10: Forces acting on the coupled pendulum.

The third step is to substitute the condition for normal modes: $x_1 = C_1 \cos\omega t$ and $x_2 = C_2 \cos\omega t$ and solve for the two ω and C_1/C_2 (there are two equations and 2 unkwnowns).

$$-\omega^2 C_1 + (\omega_0^2 + \omega_s^2) C_1 - \omega_s^2 C_2 = 0 \tag{104}$$
$$-\omega^2 C_2 + (\omega_0^2 + \omega_s^2) C_2 - \omega_s^2 C_1 = 0 \tag{105}$$
$$\Rightarrow \tag{106}$$
$$\frac{C_1}{C_2} = \frac{\omega_s^2}{-\omega^2 + (\omega_0^2 + \omega_s^2)} \tag{107}$$
$$\frac{C_1}{C_2} = \frac{-\omega^2 + (\omega_0^2 + \omega_s^2)}{\omega_s^2} \tag{108}$$
$$\Rightarrow \tag{109}$$
$$\omega_s^4 = \left(-\omega^2 + \omega_0^2 + \omega_s^2\right)^2 \tag{110}$$
$$\Rightarrow \tag{111}$$
$$\omega^2 = \omega_0^2 + \omega_s^2 \mp \omega_s^2 \tag{112}$$

so the two solutions are: $\omega_- = \omega_0$ and $\omega_+ = \sqrt{\omega_0^2 + 2\omega_s^2}$. Solving for C_1/C_2 gives:

$$\left.\frac{C_1}{C_2}\right|_- = \frac{\omega_s^2}{-\omega_0^2 + \omega_0^2 + \omega_s^2} = +1 \tag{113}$$
$$\left.\frac{C_1}{C_2}\right|_+ = -1 \tag{114}$$
$$\tag{115}$$

4.4 Double and triple pendulums: pictorial view

Figure 11 gives a sketch of the normal modes in double and triple pendulums and for a system of three masses connected with four springs of equal constant k.

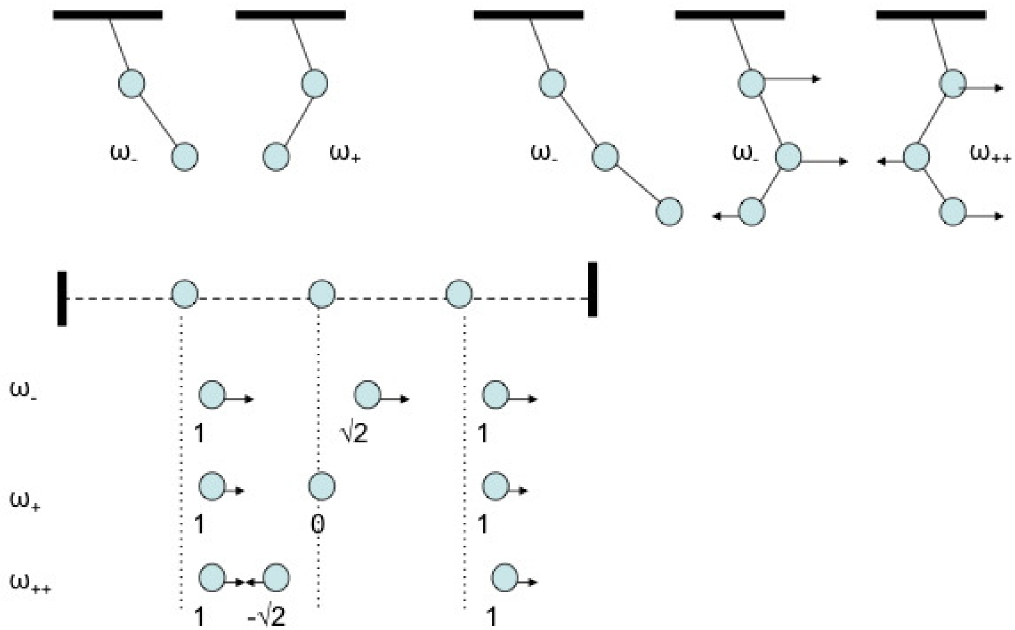


Figure 11: Normal modes of double and triple pendulums and a system of three masses connected with four springs of equal constant k.

4.5 The double pendulum

Consider a double pendulum consisting of two equal masses m attached to strings of equal length ℓ. This system has two normal modes, whose frequencies can be calculated by applying the general recipe introduced earlier. By considering all forces, in the small angle approximation $T_1 \approx 2mg$ and $T_2 \approx mg$, so the dynamical equation for the mass number 2 is:

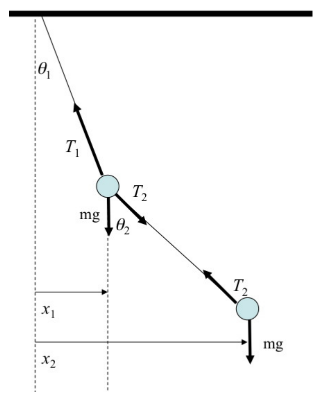


Figure 12: The double pendulum.

$$m_2\ddot{x}_2 = -T_2\sin\theta_2 = -\frac{mg}{\ell}\theta_2 = -\frac{mg}{\ell}(x_2 - x_1) \tag{116}$$

$$\Rightarrow \ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0 \tag{117}$$

where $\omega_0^2 = g/\ell$. For the mass number 1:

$$m\ddot{x}_1 = -T_1\sin\theta_1 + T_2\sin\theta_2 = -2mg\frac{x_1}{\ell} + mg\frac{x_2 - x_1}{\ell} \tag{118}$$

$$\Rightarrow \ddot{x}_1 + 2\omega_0^2 x_1 + \omega_0^2 x_1 - \omega_0^2 x_2 = 0 \tag{119}$$

$$\Rightarrow \ddot{x_1} + 3\omega_0^2 x_1 - \omega_0^2 x_2 = 0 \tag{120}$$

We can now impose the normal modes solutions: $x_1 = C_1\cos\omega t$, $x_2 = C_2\cos\omega t$, and substitue into the equations gives:

$$\begin{cases} -C_1\omega^2 + 3\omega_0^2 C_1 - \omega_0^2 C_2 = 0 \\ -C_2\omega^2 + \omega_0^2 C_2 - \omega_0^2 C_1 = 0 \end{cases} \tag{121}$$

this can be easily solved with Cramer's rule:

$$\Delta = \begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{vmatrix} \tag{122}$$

and

$$C_1 = \frac{\begin{vmatrix} 0 & -\omega_0^2 \\ 0 & -\omega^2 + \omega_0^2 \end{vmatrix}}{\Delta} \tag{123}$$

$$C_2 = \frac{\begin{vmatrix} 2\omega_0^2 - \omega^2 & 0 \\ -\omega_0^2 & 0 \end{vmatrix}}{\Delta} \tag{124}$$

Notice that the determinant is zero in both the numerators for C_1 and C_2, but since $C_{1,2} \neq 0$ (trivial solution), it follows that Δ must be 0:

$$\left(3\omega_0^2 - \omega^2\right) \cdot \left(-\omega^2 + \omega_0^2\right) - \omega_0^4 = 0 \tag{125}$$

$$\Rightarrow \omega^4 + \omega^2(-4\omega_0^2) + 2\omega_0^4 = 0 \tag{126}$$

$$\Rightarrow \omega^2 = \omega_0^2(2 \pm \sqrt{2}) \Rightarrow \begin{cases} \omega_-^2 = (2 - \sqrt{2})\omega_0^2 \\ \omega_+^2 = (2 + \sqrt{2})\omega_0^2 \end{cases} \tag{127}$$

substitute ω in the previous equations to determine C_1/C_2 for the two normal modes:

$$\left.\frac{C_2}{C_1}\right|_- = 1 + \sqrt{2} \qquad \left.\frac{C_2}{C_1}\right|_+ = \frac{-1}{1 + \sqrt{2}} \tag{128}$$

Each of these two normal modes is a solution to the double pendulum, therefore any linear superposition of the two is also a solution; the initial conditions will determine the particular linear superposition.

4.6 Driven double pendulum

We can drive a double pendulum by oscillating the tip of the string with a displacement $\eta = \eta_0 \cos\omega t$ (see figure). The equations of motion are derived similarly to the simple double pendulum case, with the exception that:

$$\sin\theta_1 = \frac{x_1 - \eta}{\ell} \tag{129}$$

which leads to the following equation for the mass 1:

$$m\ddot{x}_1 = -2mg\frac{x_1 - \eta}{\ell} + mg\frac{x_2 - x_1}{\ell} \tag{130}$$

$$\Rightarrow \ddot{x}_1 + 3\omega_0^2 x_1 - \omega_0^2 x_2 = 2\omega_0^2 \eta_0 \cos\omega t \tag{131}$$

while the equation for mass 2 is the same as for the undriven case. After imposing the normal modes conditions (ω is no longer an unknown, it is determined by the driver frequency), we can find the steady state solutions determining C_1 and C_2:

$$\begin{cases} -C_1\omega^2 + 3\omega_0^2 C_1 - \omega_0^2 C_2 = 2\omega_0^2 \eta_0 \\ -\omega_0^2 C_1 - C_2\omega^2 + \omega_0^2 C_2 = 0 \end{cases} \tag{132}$$

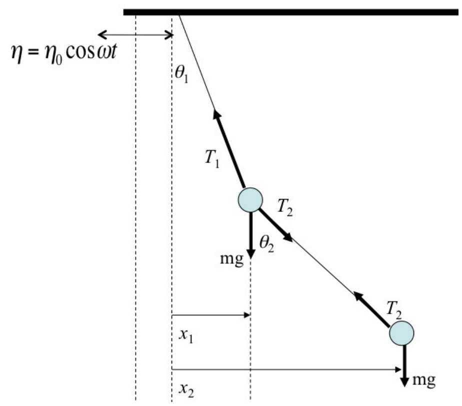


Figure 13: The driven double pendulum.

$$C_1 = \frac{\begin{vmatrix} 2\omega_0^2\eta_0 & -\omega_0^2 \\ 0 & -\omega^2 + \omega_0^2 \end{vmatrix}}{\Delta} \tag{133}$$

$$C_2 = \frac{\begin{vmatrix} 2\omega_0^2 - \omega^2 & 2\omega_0^2\eta_0 \\ -\omega_0^2 & 0 \end{vmatrix}}{\Delta} \tag{134}$$

$$C_1 = \frac{(2\omega_0^2\eta_0)(\omega_0^2 - \omega^2)}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)} \qquad C_2 = \frac{2\omega_0^4\eta_0}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)} \tag{135}$$

where ω_- and ω_+ are the solutions to $\Delta = 0$, i.e. the resonance frequencies. Note that $C_{1,2} \propto \eta_0$, and are functions of the driver frequency ω. For $\omega \to \omega_-$ we have $C_2/C_1 \to 2.4$, for $\omega \simeq \omega_0$ $C_1 \to 0$ (the upper mass stops), and for $\omega \to \omega_+$ $C_1/C_2 \to -2.4$. The figure below shows the amplitues of the masses in the driven double pendulum as a function of the driving angular frequency. Negative values of C indicate out-of-phase with the driver.

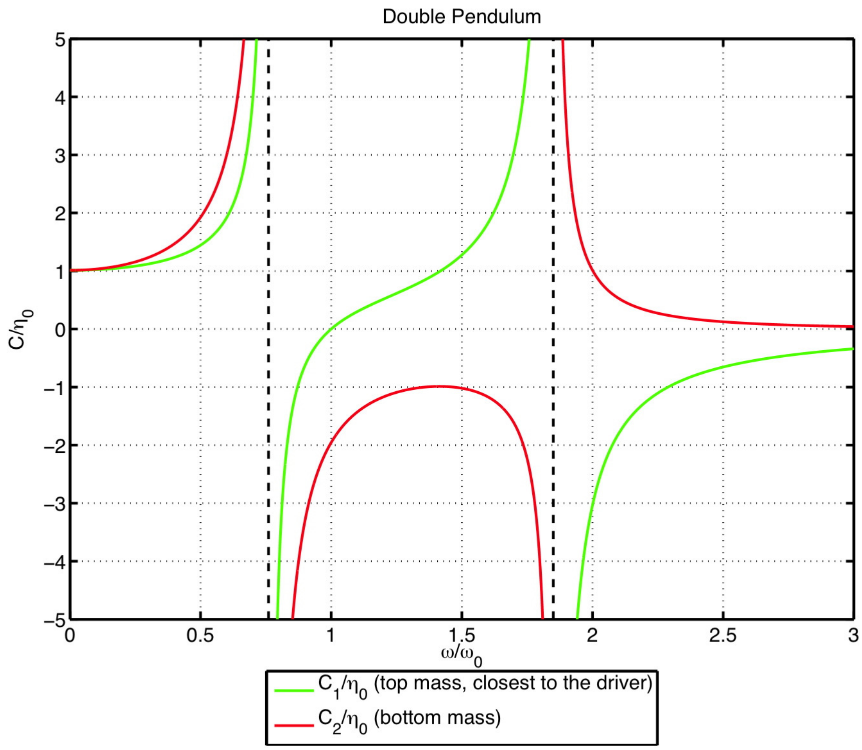
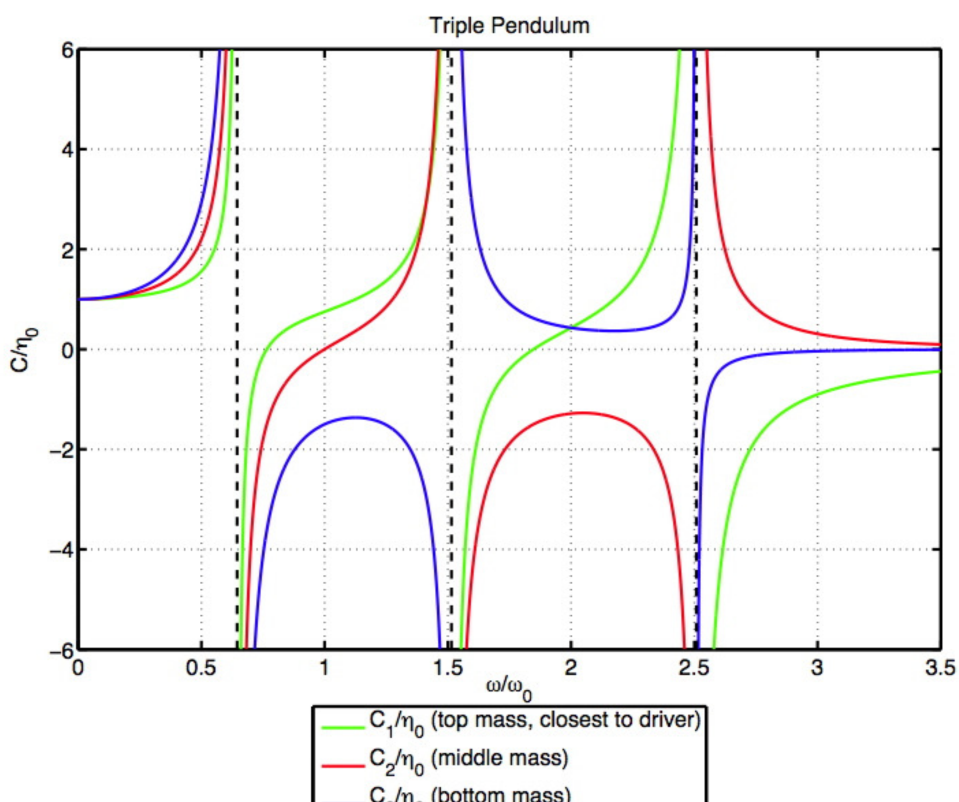


Figure 14: Solutions to the driven double pendulum.



4.7 Many coupled oscillators

We are now going to couple a large number N of oscillators, and then take the limit $N \to \infty$. Consider a string with N beads attached at regular intervals ℓ through a massless string of length $(N+1)\ell$ (see Figure). The beads may oscillate with small amplitude in the direction transverse to the string (up and down). After some arbitrary time t, the beads are not in their equilibrium position, but the end points are constrained $(y(x = 0\ell) = 0$ and $y(x = [N+1]\ell) = 0$.

Figure 16: N coupled oscillators.

Consider the small segment of string comprising the p^{th} bead and the two beads either side of it: the $(p-1)^{th}$ bead and the $(p+1)^{th}$ bead. Let us assume also (in order to simplify the algebra) that for modest amplitudes: the tension T on the string remains the same, and that there is no motion of the beads along the direction parallel to the string line, so only forces perpendicular to the string line can be considered.

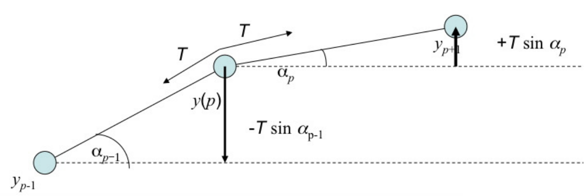


Figure 17: Forces acting on the $p-th$ bead of the string.

We can write Newton's second law for the $p-th$ bead as:

$$m\ddot{y} = -T\sin\alpha_{p-1} + T\sin\alpha_p \tag{136}$$

$$\Rightarrow m\ddot{y} = -T\frac{y_p - y_{p-1}}{\ell} + T\frac{y_{p+1} - y_p}{\ell} \tag{137}$$

and introducing the notation: $\omega_0^2 = T/(m\ell)$ the equation becomes:

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p-1} + y_{p+1}) = 0 \tag{138}$$

with the boundary conditions $y_0 = 0$ and $y_{N+1} = 0$. We now need to find solutions to this equation.
It is simpler to see the problem with only 2 beads ($p = 2$), and then we can extend the solution. For $p = 2$, the equations for the two beads are:

$$\begin{cases} \ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2 (y_2 + y_0) = 0 \\ \ddot{y}_2 + 2\omega_0^2 y_2 - \omega_0^2 (y_3 + y_1) = 0 \end{cases} \tag{139}$$

and $y_3 = 0$ and $y_0 = 0$ due to the boundary conditions. We can look for the normal modes, and use the trial function: $y_p = A_p \cos(\omega t)$, where ω is the normal mode frequency and A_p the amplitudes. The two normal modes are shown in figure below. substituting the normal mode solutions we get:

$$\begin{cases} -\omega^2 A_1 + 2\omega_0^2 A_1 - \omega_0^2 (A_2 + A_0) = 0 \\ -\omega^2 A_2 + 2\omega_0^2 A_2 - \omega_0^2 (A_3 + A_1) = 0 \\ .... \\ -\omega^2 A_p + 2\omega_0^2 A_p - \omega_0^2 (A_{p+1} + A_{p-1}) = 0 \ (generalising\ to\ the\ p^{th}\ beads) \end{cases} \tag{140}$$

This equation for $p = 1, .., N$ corresponds to N differential equations that must be solved. We will call on intuition for a few simple cases. For one bead, there is one normal mode, and the vibration looks like the one in figure below:

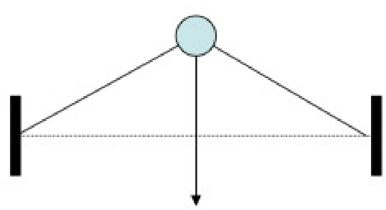


Figure 18: Normal modes of 1 bead attached to a string.

For two beads, there are two normal modes, where the beads oscillate with simple harmonic motion either in phase or 180° out-of-phase in the two normal modes respectively (see figure).

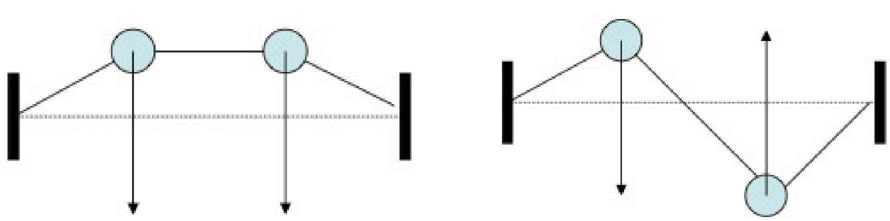


Figure 19: Normal modes of two beads attached by a string.

These shapes start looking familiar, as the lowest frequency mode for each number of beads looks like a sinusoidal function over hald a cycle; the next lowest mode looks like a sinusoidal function over a full cycle. For three beads, there are three normal modes, which can be drawn by drawing the fundamental, first and second harmonic, and then placing the beads equally spaced and connected.

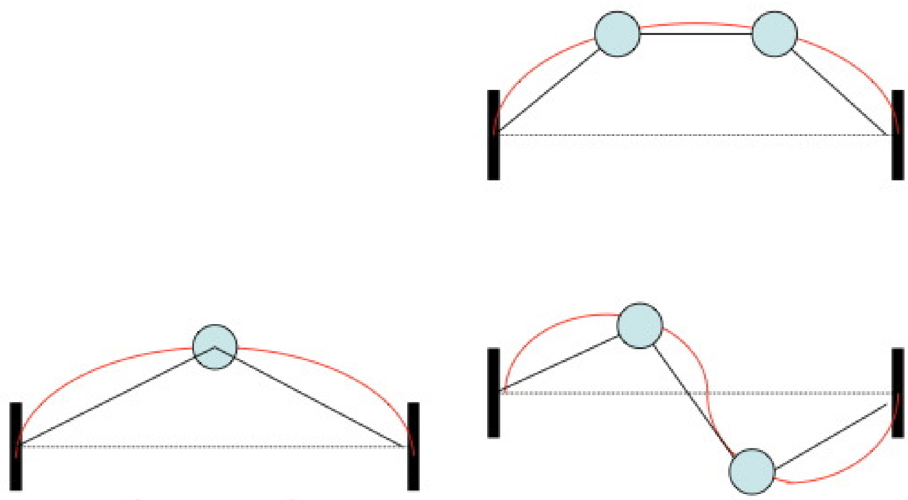


Figure 20: Normal modes of one and two beads attached by a string.

Without proving it, the solution for $A_{p,n}$ is (where p is the beads index and n indicate one of the N normal modes):

$$A_{p,n} = C_n \sin\left(\frac{pn\pi}{N+1}\right) \tag{141}$$

note that for $p = 0$ we get $A_0 = 0$, and for $p = N+1$ also $A_{N+1} = 0$ so the solution satisfies the boundary conditions. If we take $n = 1, 2$:

$$n = 1 \Rightarrow A_{p,1} = C_1 \sin\left(\frac{p\pi}{N+1}\right) \tag{142}$$

$$n = 2 \Rightarrow A_{p,2} = C_2 \sin\left(\frac{2p\pi}{N+1}\right) \tag{143}$$

$$\tag{144}$$

These are the solutions for the amplitudes. We also need to determine the actual frequencies of the normal modes, for which we go back to the N beads equations, substitute the solutions for $A_{n,p}$ and get:

$$\omega_n = 2\omega_0 \sin\frac{n\pi}{2(N+1)} \tag{145}$$

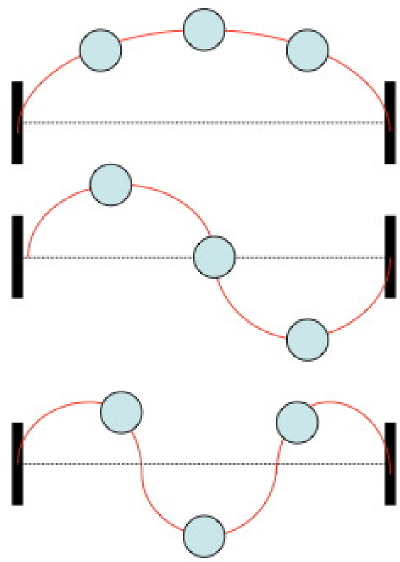


Figure 21: Normal modes of three beads attached by a string.

The general solution for the motion of N beads attached to a string will be the linear superposition of the N normal modes, with the actual amplitudes given by the initial conditions.

5 The wave equation

We can imagine a string as the limit $N \to \infty$ as the number of beads gets infinitely large and their separation ℓ gets small ($N \to \infty$ and $m, \ell \to 0$. We can think of the beads as the atoms in a string. The distribution of points in the normal modes of oscillation starts to look almost continuous. Looking at the p^{th} beads, now:

$$\frac{y_{p+1} - y_p}{\ell} \to \left.\frac{dy}{dx}\right|_{x=x_{p+1}} \quad \text{and} \quad \frac{y_p - y_{p-1}}{\ell} \to \left.\frac{dy}{dx}\right|_{x=x_p} \tag{146}$$

so the equation becomes:

$$\frac{d^2y}{dt^2} = \frac{T}{m}\left[\left.\frac{dy}{dx}\right|_{x=x_p} - \left.\frac{dy}{dx}\right|_{x=x_{p+1}}\right] \tag{147}$$

and calling $m/\ell = \mu$, and taking $\ell \to 0$

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{1}{\ell}\left[\left.\frac{dy}{dx}\right|_{x=x_p} - \left.\frac{dy}{dx}\right|_{x=x_{p+1}}\right] \tag{148}$$

$$\Rightarrow \frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2} \tag{149}$$

$$\tag{150}$$

note that the dimension of $T\mu = [v^2]$, so indicating with $v = \sqrt{T/\mu}$ the equation can be written as:

$$\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{d^2y}{dt^2}\frac{1}{v^2} - \frac{d^2y}{dx^2} = 0 \tag{151}$$

5.1 Pulse propagation

Any function $f(x \pm ct)$, with $c \in \mathbb{R}$, is a solution to the wave equation. This function represents a travelling function, in the positive or negative direction depending on the $\pm$ sign. If this is a pulse propagating onto a string, the velocity of the propagation will be $c = \sqrt{T/\mu}$ so that a thicker cord (higher density) will give slower propagation speeds.

5.2 Normal modes of a string with fixed ends

We want to look for the normal mode solutions of a string fixed at both ends. One way of solving the problem is to look back at the results for N beads attached to a string, and then take the $N \to \infty$ limit (see A.P. French for such derivation). We will take a different approach, but superimposing two waves travelling in opposite directions and then close the ends of the string.

Travelling disturbances and waves

We have seen in the previous section how any function $f(x \pm vt)$ is a solution to the wave equation, and describes a travelling function. Suppose we now generate waves by moving the end of the string up and down with angular frequency ω and amplitude A, the function which I generate is of form:

$$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \tag{152}$$

This is the matematical description of a travelling sinusoidal disturbance, $v = \sqrt{T/\mu}$ is the velocity, given by the string, and λ is the distance the perturbance travels in one oscillation time (wavelength), determined by ω and v. Note that for $t = 0$:

$$y(x,0) = A\sin\left(\frac{2\pi}{\lambda}x\right) \tag{153}$$

and for $x = 0 \;\Rightarrow\; y(0,0) = 0$ and $x = \lambda \;\Rightarrow\; y(\lambda, 0) = 0$. The eq. (152) is clearly a solution to the wave equation, and we call v the **phase verlocity**, i.e. the velocity of a point of constant phase.

We can rewrite eq. (152) in a different form by defining: $k = (2\pi)/\lambda$ (**wave number**), and since, if P is the period of the oscillation:

$$\lambda = vP = v\frac{2\pi}{\omega} \;\Rightarrow\; \frac{2\pi v}{\lambda} = \omega \;\Rightarrow kv = \omega \tag{154}$$

we can then rewrite the sinusoidal travelling wave as:

$$y(x,t) = A\sin(kx - \omega t) \tag{155}$$

Superposition of oppositely-travelling sinusoidal waves

We now generate a sinusoidal wave travelling in opposite direction:

$$y(x,t) = A\sin(kx + \omega t) \tag{156}$$

and let us sum the two:

$$y(x,t) = y_1 + y_2 = A\sin(kx - \omega t) + A\sin(kx + \omega t) = 2A\sin(kx)\cos(\omega t) \tag{157}$$

note that the first term contains the spacial information and the second contains the time information, this is called a **standing wave**. As a final step let us fix the string at both ends: $y(x,t) = 0$ at $x = 0$ and $x = L$, which implies:

$$\sin(kL) = 0 \;\Rightarrow\; k_n L = n\pi \;\Rightarrow\; k_n = \frac{n\pi}{L} \tag{158}$$

therefore requesting the boundary conditions of fixed ends, causes that only discrete values of k_n are allowed. Conditions on the wave numbers correspond also to:

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}; \qquad \omega_n = k_n v = \frac{n\pi v}{L}; \qquad f_n = \frac{\omega_n}{2\pi} = \frac{nv}{2L} \tag{159}$$

These solutions for $n = 1, .., \infty$ correspond to the harmonics. Note also that the frequency of the harmonics is n times the fundamental ($n = 1$). The $n-th$ harmonic of a string with fixed ends is described by the solution:

$$y(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) \tag{160}$$

and the generic solution to the wave equation for a string with fixed ends will be a linear superposition with various values of A_n of these n harmonics.

5.3 Longitudinal waves (sound)

In longitudinal waves, the motion of the particles is along the axis of the wave propagation, and an example is sound waves. Similarly to a string with fixed ends, we can write normal modes for a sound wave in a tube of length L and closed ends; indicating with ξ the displacement of the molecules, the normal modes are:

$$\xi_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) \tag{161}$$

which is identical to the normal mode for a string with closed ends but with the v now given by the speed of sound ($\sim$340 m/s in air). It is more often found that sound waves are expressed in terms of over-pressure (pressure above the atmosphere), in which case the center of the tube, which would be a node if expressed in terms of displacement, becomes an anti-node in terms of over-pressure:

$$P_n(x,t) = C_n \cos\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) \tag{162}$$

5.4 Energy of a wave

Consider a string with a sinusoidal wave travelling: $y(x,t) = A\sin[k(x - vt)]$. No matter is being moved forward in the x direction but particles with a certain mass are moving in the y direction, and they have kinetic energy; the kinetic energy of a section dx is:

$$dk_E = \frac{1}{2}(dm)v_y^2 = \frac{1}{2}\mu dx \left(\frac{\partial y}{\partial t}\right)^2 \tag{163}$$

where v_y is the transverse velocity of the string's particles, not to confuse with the phase velocity of the wave. The value of $\partial y/\partial t$ can be easily calculated from the sinusoidal wave function, and by integrating over 1 wavelegth we obtain:

$$E_k = \frac{1}{2}\mu A^2 k^2 v^2 \int_0^\lambda \cos^2[k(x - vt)] = \frac{A^2\pi^2 T}{\lambda} \tag{164}$$

as the integral above is $\lambda/2$. The string has also potential energy, which is the work that needs to be done to create the distorced shape on the string, and it can be shown to be equal to the kinetic term above, infact, the potential energy of a section dS of a string with tension T is $d_{PE} = T(dS - dx)$, where dx is unstretched, and:

$$dS = \sqrt{dx^2 + dy^2} = dx\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2} \simeq dx\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right] \tag{165}$$

giving:

$$dPE = T\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx \tag{166}$$

which is equal to the kinetic energy calculated above. Therefore the total energy is:

$$E_{Tot} = \frac{2A^2\pi^2 T}{\lambda} \tag{167}$$

Since a standing wave of amplitude A can be created by superimposing two oppositely travelling waves with amplitudes $A/2$, it is immediate to see that a standing wave has energy:

$$E_{Tot} = \frac{A^2\pi^2 T}{\lambda} \tag{168}$$

i.e. half the energy of a travelling wave of the same amplitude. The power needed to generate a travelling wave can be calculated as:

$$P = \frac{2A^2\pi^2 T}{\lambda}\frac{1}{T} = \frac{2A^2\pi^2}{\lambda} \tag{169}$$

5.5 Wave transmission

Consider two strings with different mass densities μ_1 and μ_2 attached at $x = 0$; if we send a incident wave along the string 1, there will be a reflected wave on 1 and a transmitted wave on 2. The boundary conditions are that:

$$
\begin{aligned}
y_1 &= y_2 \qquad \text{for } x = 0 && (170)\\
\frac{\partial y_1}{\partial x} &= \frac{\partial y_2}{\partial x} \qquad \text{for } x = 0 && (171)\\
& && (172)
\end{aligned}
$$

the first condition corresponds to the fact that the two strings are attached, whereas the second condition requires that there can never be a kink at the junction between the two strings. If:

$$y_i = A_i \sin(\omega_1 t - k_1 x) \tag{173}$$

is the incident wave, then the reflected wave will be:

$$y_r = A_r \sin(\omega_1 t + k_1 x) \tag{174}$$

with the same waveleght and frequency. There will also be a transmitted wave however:

$$y_{tr} = A_{tr} \sin(\omega_1 t - k_2 x) \tag{175}$$

note that the frequency ω_1 of the transmitted wave is the same as the incident wave, since the junction shakes with the same frequency: $\omega_1 = v_1 k_1 = v_2 k_2$. However, the wave number k_2 is different for the transmitted wave. We can also calculate the amplitudes, using the boundary conditions:

$$\begin{cases} A_i + A_r = A_{tr} \\ -A_i k_1 + A_r k_1 = -k_2 A_{tr} \end{cases} \tag{176}$$

and using $k_1/k_2 = v_2/v_1$ we get:

$$
\begin{aligned}
\frac{A_r}{A_i} &= \frac{v_2 - v_1}{v_2 + v_1} \qquad \text{Reflectivity} && (177)\\
\frac{A_{tr}}{A_i} &= \frac{2v_2}{v_1 + v_2} \qquad \text{Transmittivity} && (178)\\
& && (179)
\end{aligned}
$$

Note that a closed end can be thought as $\mu_2 \to \infty$, hence $v_2 = 0$ which gives $R = -1$ and $Tr = 0$ (fully reflected wave, upsidedown). For $\mu_1 > \mu_2$, $v_1 < v_2$, $R > 0$ and $Tr > 0$.

5.6 Wave dispersion

When the speed of propagation is independent of the frequency (or the wavelenght), we call the media a *non-dispersive* media. On the other hand, a wave propagating through a medium for which the speed of propagation is dependent on the frequency is propagating through what is called a *dispersive* medium.

An example of dispersive media: beads on a string

Revisit the case of N beads on a string, with beads of mass m, spaced a distance ℓ on a string of total length L. We have found that the general result for the frequency:

$$\omega_n = 2\omega_0 \sin\frac{n\pi}{2(N+1)} \tag{180}$$

and $\omega_0 = \sqrt{T/(m\ell)}$. We have also found that:

$$k_n = \frac{n\pi}{L} \tag{181}$$

so the speed of propagation: $v = \omega_n/k_n$ is cleary not independent on n, since k_n is linear on n while $\omega_n \propto \sin(n)$. This is an example of dispersion. A plot of ω_n versus k_n would show that the speed of propagation is lower for higher frequencies, given by higher n.

Consequences of dispersion

Consider the superposition of two waves travelling on a dispersive media; let the first wave have wave number and frequency k_1, ω_1 and the second wave wave number and frequency k_2, ω_2; their velocity of propagation will be $v_1 = \omega_1/k_1$ and $v_2 = \omega_2/k_2$, $v_1 \neq v_2$:

$$y = y_1 + y_2 = A\sin(k_1 x - \omega_1 t) + A\sin(k_2 x - \omega_2 t) = \tag{182}$$

$$= 2A\sin\left[\frac{(k_1+k_2)}{2}x - \frac{(\omega_1+\omega_2)}{2}t\right]\cos\left[\frac{(k_1-k_2)}{2}x - \frac{(\omega_1-\omega_2)}{2}t\right] \tag{183}$$

If the two travelling waves have wave numbers and frequencies not too dissimilar: $k_1 \approx k_2$ and $\omega_1 \approx \omega_2$, then we can indicate with $k = (k_1+k_2)/2$ and $\omega = (\omega_1+\omega_2)/2$, and the total of the two waves above can be written as:

$$y = 2A\sin\left[kx - \omega t\right]\cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right] \tag{184}$$

The sin term is a travelling wave with velocity $v_p = \omega/k$ called *phase velocity*, and has a wave-length $2\pi/k$; the cos term defines an envelope, itself moving with velocity $v_g = \Delta\omega/\Delta k$ called *group velocity*, and wave-lenght $4\pi/\Delta k$.

This effect is more general than just for the case of two sinusoidal waves: we always define the phase velocity as the ratio $v_p = \omega/k$, and the group velocity as $v_g = d\omega/dk$. The group velocity can be faster or slower, or even of opposite sign than the phase velocity: media for which $v_g < v_p$ are called *normally dispersive*, while $v_g > v_p$ are called *anomalously dispersive*. The ω vs. k curve is called *dispersion relation*.

Dispersion in a string

In deriving the wave equation for a string we found:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad \text{with } v = \sqrt{\frac{T}{\mu}} \tag{185}$$

clearly v is a constant, independent of ω and therefore such string is non-dispersive, and its dispersion relation is:

$$\omega = vk \tag{186}$$

However, in deriving the equation above for the string, we only considered the tension T as responsible exclusively for the restoring force. In reality, stiffness of the wire will contribute, so that the equation becomes:

$$\frac{\partial^2 y}{\partial x^2} - \alpha\frac{\partial^4 y}{\partial x^4} - \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0 \tag{187}$$

The solution is still of the form $y(x,t) = A\sin(kx - \omega t)$ but:

$$\omega^2 = v_0^2 k^2 [1 + \alpha k^2] \tag{188}$$

where $v_0^2 = T/\mu$. The phase velocity is therefore:

$$v_p = \frac{\omega}{k} = v_0\sqrt{(1 + \alpha k^2)} \tag{189}$$

and since $v_p > v_0$, stiffness makes the wave go faster. Also note that the group velocity is:

$$v_g = \frac{d\omega}{dk} = v_p + v_0\frac{\alpha k}{\sqrt{1 + \alpha k^2}} \tag{190}$$

As an example, piano strings are usually stiff and therefore dispersive.

5.7 Water waves

Deep water: waves and ripples

When water is much deeper than the waveleght of the wave propagating on its surface, the dispersion relation is:

$$\omega^2 = gk + \frac{s}{\rho}k^3, \tag{191}$$

where g is the gravitational acceleration, s is the surface tension (~ 0.072N/m, ρ is the water density ($\sim 10^3$kg/m^3). If $\lambda >> 1$cm, then the s term can be neglected, and:

$$\omega \simeq \sqrt{gk} \qquad v_p = \frac{\omega}{k} = \sqrt{g}k \propto \sqrt{\lambda} \tag{192}$$

$$\text{and } v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{g/k} = \frac{1}{2}v_p \tag{193}$$

therefore the higher the wavelenght, the faster the wave.
For $\lambda << 1$cm (ripples), the s term in the dispersion relation completely dominates:

$$\omega^2 \simeq \frac{s}{\rho}k^3 \qquad v_p = \frac{\omega}{k} = \sqrt{\frac{s}{\rho}k} \propto \frac{1}{\sqrt{\lambda}} \tag{194}$$

$$\text{and } v_g = 1.5v_p \tag{195}$$

Shallow water: waves and ripples

The full dispersion relation for water waves is:

$$\omega^2 = \left(gk + \frac{s}{\rho}k^3\right)\tanh(kh), \tag{196}$$

where h is the depth of the water. Of course for deep water waves, $\lambda << h$ and so $\tan(kh) \approx 1$, which gives the relation seen earlier, but for $\lambda >> h$ then $\tan(kh) \approx kh$ and the equation:

$$\omega^2 \simeq \left(gk + \frac{s}{\rho}k^3\right)(kh) = gk^2h + \frac{k^4 s}{\rho}h \tag{197}$$

For large waves, $\lambda > 1$cm, the surface term is negligible, and the dispersion relation becomes simply:

$$\omega^2 \simeq gk^2\lambda \;\Rightarrow\; \omega = k\sqrt{gh} \tag{198}$$

$$\text{and} v_p = \frac{\omega}{k} = \sqrt{gh} \tag{199}$$

note that it is independent on λ (non-dispersive), but it crucially depends on h, creating a *tidal* wave: as it approaches the shore, v_p decreases and the wave builds up, creating a tsunami-wave.