BSc/MSci MidTerm Test

PHY-217 Vibrations and Waves

Time Allowed: 40 minutes

Date: 18^{th} Nov, 2011

Time: 9:10 - 9:50

Instructions: **Answer ALL questions in section A. Answer ONLY ONE questions from section B. Section A carries 25 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.**

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiner: Dr L.Cerrito

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SECTION A. Attempt answers to all questions.

A1. For a particle of mass m executing simple harmonic motion ("SHM"), its displacement from equilibrium can be written as:

$$x(t) = Asin(\omega_0 t + \phi).$$

(i) Write down the amplitude, angular frequency, period and phase at time t. **[3]**

(ii) Sketch a graph of $x(t)$. **[2]**

SOLUTION:
(i) amplitude: A, angular frequency: ω_0, period: $T = 2\pi/\omega_0$, phase: $\omega_0 t + \phi$
(ii) sinusoidal graph

A2. A mass m is attached to a massless spring with constant k, and can move horizontally on a frictionless plane without air resistance.

(i) By explicit consideration of the forces involved, derive the differential equation of motion and write down the general solution. **[2]**

(ii) What is the angular frequency, expressed in terms of k and m? **[2]**

(iii) Write down the potential, kinetic, and total energy at time t. **[2]**

(iv) Write down an example of initial conditions. **[1]**

SOLUTION:
(i) The equation is: $m\ddot{x} = -kx \Rightarrow \ddot{x} + \omega_0^2 x = 0$ with $\omega_0^2 = k/m$
general solution: $x(t) = Acos(\omega_0 t + \phi)$ with A, ϕ constants.
(ii) $\omega_0^2 = k/m$
(iii) $E_p(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kA^2cos^2(\omega_0 t + \phi)$ and $E_k(t) = \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}m\omega_0^2 A^2 sin^2(\omega_0 t + \phi)$
Total energy: $E_k + E_p = \frac{1}{2}kA^2$
(iv) An example is: $x(t = 0) = x_0$ and $v(t = 0) = v_0$.

A3. Show that the function $y = Acos(4x) + Bsin(4x)$, where A and B are arbitrary constants, is a general solution of the differential equation: **[4]**

$$\frac{d^2y}{dx^2} + 16y = 0.$$

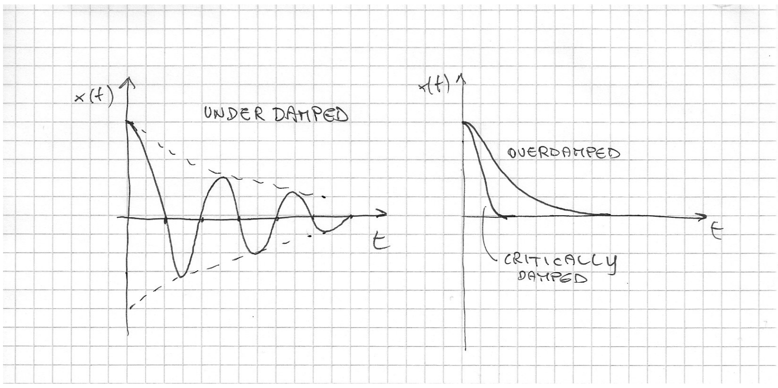
SOLUTION:

$$\frac{dy}{dx} = -4Asin(4x) + 4Bcos(4x) \quad \text{and} \quad \frac{d^2y}{dx^2} = -16Acos(4x) - 16Bsin(4x)$$

$$\Rightarrow \frac{d^2y}{dx^2} + 16y = 0$$

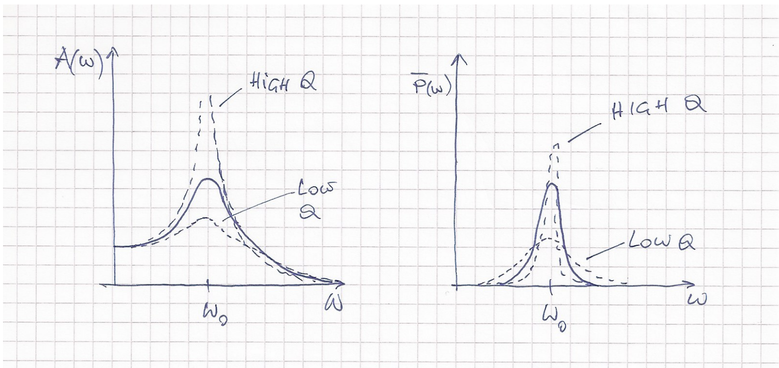
A4. For an object starting from maximum diplacement and released from rest, sketch plots of displacement versus time for: (i) underdamped SHM, (ii) critically damped SHM and (iii) overdamped SHM. (iv) How is the oscillator's Quality Factor, Q, defined? **[4]**

SOLUTION:
Q is a dimensionless number defined as: $Q = \frac{1/\gamma}{1/\omega_0} = \omega_0/\gamma$, where γ is the damping constant in damped harmonic motion and ω_0 is the angular frequency of the undamped oscillator.



A5. Consider a damped driven simple harmonic oscillator in steady state. Sketch plots of (i) amplitude versus driving frequency, and (ii) average power versus driving frequency. Indicate on each plot the resonance frequency and what effect high or low oscillator's quality factor, Q, has on the graph. **[3]**

SOLUTION:



A6. Consider N equal masses coupled by springs: how many normal modes does such a system have? For a normal mode, what characterises the motion of each of the masses? **[2]**

SOLUTION:
A system of N masses coupled by springs has N normal modes. In normal mode vibration all masses oscillate with the same frequency and are either in phase or 180° out-of-phase.SOLUTION:
A system of N masses coupled by springs has N normal modes. In normal mode vibration all masses oscillate with the same frequency and are either in phase or 180° out-of-phase.

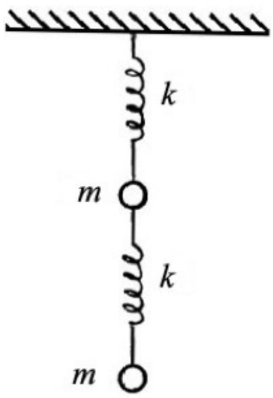
SECTION B. Answer one of the two questions in this section.

B1. Two equal masses are connected as shown with two identical massless springs of spring constant k.

(i) By explicit consideration of the forces and considering only motion in the vertical direction, derive the equations of motion of the two masses. Do you need to consider gravitational forces? **[10]**

(ii) Show that the angular frequencies of the two normal modes are given by $\omega^2 = (3\pm\sqrt{5})k/2m$ and hence that the ratio of the normal mode frequencies is $(\sqrt{5}+1)/(\sqrt{5}-1)$. **[8]**

(iii) Find the ratio of amplitudes of the two masses in each separate mode. **[7]**



SOLUTION

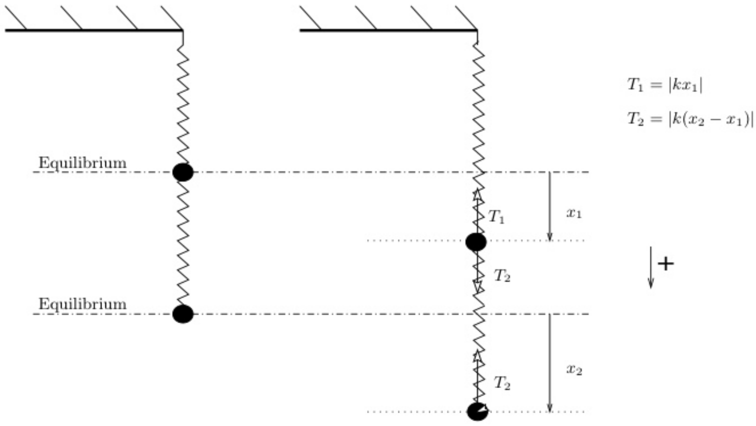
(i) Let the displacement from the equilibrium positions for masses m_1 and m_2 be x_1 and x_2 respectively. Then the tensions in the two strings are $T_1 = kx_1$ and $T_2 = k(x_2 - x_1)$ repectively. We need not to consider the gravitational forces acting on the masses, because they are independent of the displacements and hence do not contribute to the restoring forces that cause the oscillations. The gravitational forces merely cause a shift in the equilibrium positions of the masses, and you do not have to find what those shifts are. Now

$$m_1\ddot{x}_1 = +k(x_2 - x_1) - kx_1$$
$$m_2\ddot{x}_2 = -k(x_2 - x_1)$$

substituting $m_1 = m_2 = m$ and $\omega_s^2 = k/m$ we get:

$$\ddot{x}_1 = \omega_s^2(x_2 - 2x_1)$$
$$\ddot{x}_2 = \omega_s^2(x_1 - x_2)$$

(ii) Let: $x_1 = C_1\cos(\omega t)$ and $x_2 = C_2\cos(\omega t)$ and substituting above:



$$-\omega^2 C_1 + 2\omega_s^2 C_1 = \omega_s^2 C_2$$
$$-\omega^2 C_2 + \omega_s^2 C_2 = \omega_s^2 C_1$$

These can be solved directly or using Cramer's Rule. In the first case:

$$\frac{C_1}{C_2} = \frac{\omega_s^2}{2\omega_s^2 - \omega^2} = \frac{\omega_s^2 - \omega^2}{\omega_s^2}$$
$$\omega_s^4 = 2\omega_s^4 - 3\omega_s^2\omega^2 + \omega^4$$
$$\omega^4 - 3\omega_s^2\omega^2 + \omega_s^4 = 0$$
$$\omega^2 = \frac{3\omega_s^2 \pm \sqrt{9\omega_s^4 - 4\omega_s^4}}{2} = \left(3 \pm \sqrt{5}\right)\frac{\omega_s^2}{2}$$
$$\omega^2 = \left(3 \pm \sqrt{5}\right)\frac{k}{2m}$$
$$\frac{\omega_+}{\omega_-} = \sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}}} = \sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}}} = \frac{3+\sqrt{5}}{2} \cdot \frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

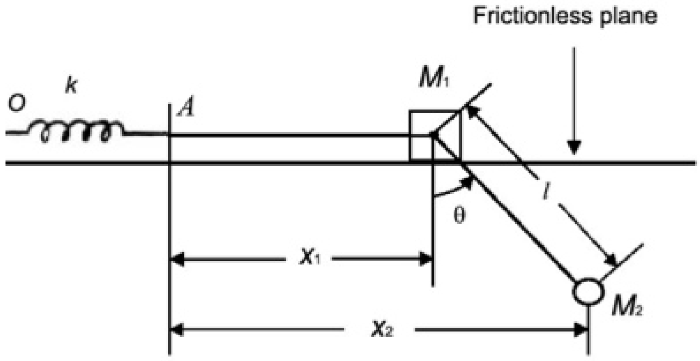
(iii) For $\omega_+ = \sqrt{(3+\sqrt{5})k/2m}$:

$$\frac{C_1}{C_2} = \frac{\omega_s^2}{2\omega_s^2 - \omega_+^2} = \frac{2\omega_s^2}{4\omega_s^2 - (3+\sqrt{5})\omega_s^2} = \frac{2}{1-\sqrt{5}}$$

For $\omega_- = \sqrt{(3-\sqrt{5})k/2m}$:

$$\frac{C_1}{C_2} = \frac{\omega_s^2}{2\omega_s^2 - \omega_-^2} = \frac{2\omega_s^2}{4\omega_s^2 - (3-\sqrt{5})\omega_s^2} = \frac{2}{1+\sqrt{5}}$$

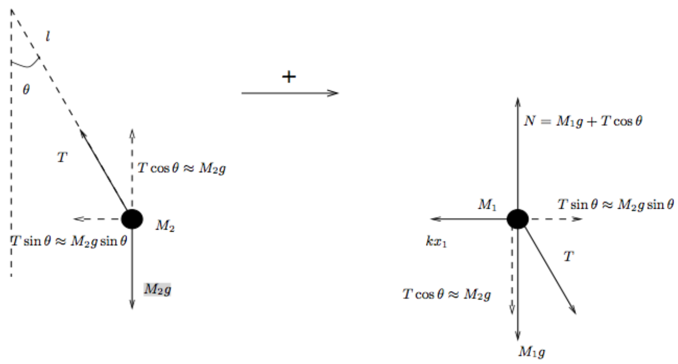
B2. The sketch shows a mass M_1 on a frictionless plane connected to support O by a spring of stiffness k. The mass M_2 is supported by a string of length l from M_1. OA is the length of the relaxed spring. x_1 and x_2 are the positions of M_1 and M_2, respectively, relative to point A.



(i) Assuming small angles θ, write down the differential equation of motion for M_1. **[8]**
(ii) Assuming small angles θ, write down the differential equation of motion for M_2. **[7]**
(iii) Now let O oscillate harmonically in the horizontal direction, driven by an external force, and its position $X(t)$ is given by $X_0\cos(\omega t)$. Let both masses be equal to M. Write down the differential equation of motion for each mass. **[5]**

(iv) What are the amplitudes (steady state) of the two masses as a function of k, M, X_0, l and ω ? (*Hint: use Cramer's rule*) **[5]**

SOLUTION



(i) The equation of motion for mass M_1 is:

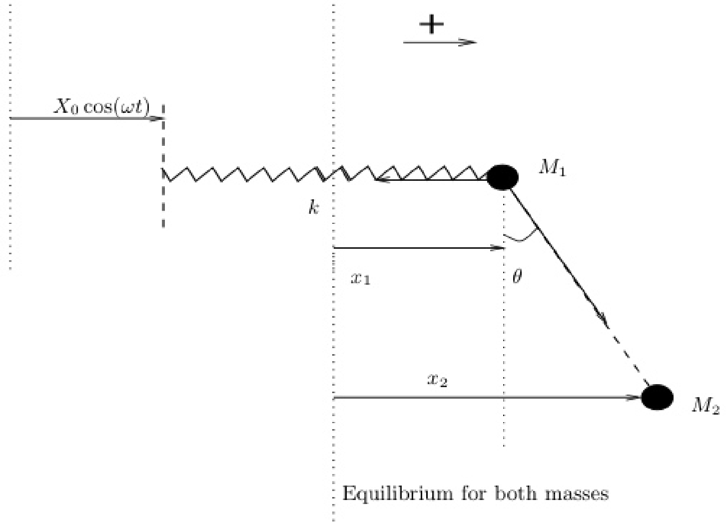
$$M_1\ddot{x}_1 = -kx_1 + M_2\frac{g}{l}(x_2 - x_1)$$

(ii) and for mass M_2 is:

$$M_2\ddot{x}_2 = -M_2 g\sin(\theta)$$
$$\Rightarrow M_2\ddot{x}_2 = -M_2\frac{g}{l}(x_2 - x_1)$$

(iii) The equation of motion for mass M_2 in unchanged, whereas for mass M_1 is:

$$M_1\ddot{x}_1 = -k[x_1 - X(t)] + M_2\frac{g}{l}(x_2 - x_1)$$
$$M_1\ddot{x}_1 + kx_1 + M_2\frac{g}{l}(x_1 - x_2) = kX_0\cos(\omega t)$$



(iv) Substituting $\omega_s^2 = k/M_2$, $\omega_p^2 = g/l$ and $M_1 = M_2 = M$ we get:

$$\ddot{x}_2 + \omega_p^2 x_2 - \omega_p^2 x_1 = 0$$
$$\ddot{x}_1 + (\omega_s^2 + \omega_p^2)x_1 - \omega_p^2 x_2 = \omega_s^2 X_0\cos(\omega t)$$

Let: $x_1 = C_1\cos(\omega t)$ and $x_2 = C_2\cos(\omega t)$, and substituting above:

$$\omega_p^2 C_1 + (\omega^2 - \omega_p^2)C_2 = 0$$
$$(-\omega^2 + \omega_s^2 + \omega_p^2)C_1 - \omega_p^2 C_2 = \omega_s^2 X_0$$

$$C_1 = \frac{\begin{vmatrix} 0 & \omega^2 - \omega_p^2 \\ \omega_s^2 X_0 & -\omega_p^2 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}}$$
$$= \frac{kX_0(g - l\omega^2)}{Ml\omega^4 - (2Mg + kl)\omega^2 + kg}$$

$$C_2 = \frac{\begin{vmatrix} \omega_p^2 & 0 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & \omega_s^2 X_0 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}}$$
$$= \frac{kgX_0}{Ml\omega^4 - (2Mg + kl)\omega^2 + kg}$$

These are the steady state solutions. The general solution is a linear combination between the transient solution and the steady state solutions.