Vibrations and waves: revision

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Form of the examination

- Part A = 50%, 10 short questions, no options
- Part B = 50%, Answer 2 questions from a choice of 4
- Total exam lasts 2.5 hours

Part A

- 10 short questions, not of equal value
- Average of 7.5 minutes per question – not much thinking time!
- Some simple definitions, including equations
- Some discussion or descriptions, including sketches
- Some simple derivations of solutions of equations

Part B

- Similar questions to the weekly practical sessions
- Some derivations of given solutions, and some calculations
- Just over half an hour per question
- Some slight twists to reward the best students, but not to worry about these

General principle

- Basic equation is force = mass x acceleration
- Force is minus the derivative of the potential energy with respect to the key spatial variable

$$E = \frac{1}{2} k x^2$$

$$F = -\frac{dE}{\mathrm{d}x} = -kx$$

$$\frac{\mathrm{d}^2 E}{\mathrm{d}x^2} = k$$

- Energy is written to the quadratic power of the key spatial variable

Simple harmonic oscillator

- Basic equation is force = mass x acceleration

$$F = -kx = ma = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\ddot{x}$$

$$-kx = m\ddot{x}$$

$$m\ddot{x}^2 + kx = 0$$

- Solution is a sinusoidal function

$$x = A\cos\omega t \quad \dot{x} = -A\omega\sin\omega t \quad \ddot{x} = -A\omega^2\cos\omega t$$

$$m\ddot{x} + kx = -Am\omega^2\cos\omega t + kA\cos\omega t = \left(-m\omega^2 + k\right)A\cos\omega t = 0$$

$$\omega^2 = k / m$$

Energy of a simple harmonic oscillator

- Kinetic + potential energy

$$
\begin{aligned}
E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\
&= \frac{1}{2}m\left(\omega A \sin\omega t\right)^2 + \frac{1}{2}k\left(A\cos\omega t\right)^2 \\
&= \frac{1}{2}A^2\left(m\omega^2\sin^2\omega t + k\cos^2\omega t\right) \\
&= \frac{1}{2}A^2\left(k\sin^2\omega t + k\cos^2\omega t\right) \\
&= \frac{1}{2}A^2 k
\end{aligned}
$$

- Average KE = average PE, total = constant

Simple examples of simple harmonic vibrations

- Oscillation of a mass on a spring
- Vibration of a simple diatomic molecule
- Simple pendulum, where we have $\omega^2 = g / L$
- Complex pendulum, where $\omega^2 = mgd / I$
- Electrical circuits, where $\omega^2 = 1 / LC$

Summary of simple harmonic vibrations

- We write energy and derive a force
- The main equation can be written in the form

$$\ddot{x} + \omega^2 x = 0$$

Simple concepts to define

- Amplitude
- Period
- Angular frequency
- Phase

Addition of damping

- We add a force (like friction) that depends on velocity

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Three regimes

- Under-damped $\omega_0^2 > \gamma^2 / 4$
- Over-damped $\omega_0^2 < \gamma^2 / 4$
- Critical damping $\omega_0^2 = \gamma^2 / 4$

Solutions

- Easiest approach is to use complex exponentials

$$x(t) = x_0 \exp(i\beta t) \quad \dot{x}(t) = x_0 i\beta \exp(i\beta t) \quad \ddot{x}(t) = -x_0 \beta^2 \exp(i\beta t)$$

- The equations follow as

$$\begin{aligned}
&\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \\
&\Rightarrow -x_0 \beta^2 \exp(i\beta t) + i x_0 \gamma\beta \exp(i\beta t) + \omega_0^2 x_0 \exp(i\beta t) = 0 \\
&\Rightarrow -\beta^2 + i\gamma\beta + \omega_0^2 = 0 \\
&\Rightarrow \beta^2 - i\gamma\beta - \omega_0^2 = 0 \\
&\Rightarrow \beta = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2} = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2 / 4}
\end{aligned}$$

Solutions

- Underdamped gives a decaying sinusoidal function, with new angular frequency $\omega^2 = \omega_0^2 - \gamma^2 / 4$

$$x(t) = A_0 \exp(-\gamma t / 2)\cos(\omega t)$$

- Overdamped gives a sum of two exponential curves

$$x(t) = \exp(-\gamma t / 2)\big(A\exp(\alpha t) + B\exp(-\alpha t)\big)$$

- Critical damping gives a decaying function that returns quickly to zero

$$x(t) = \exp(-\gamma t / 2)\big(A + Bt\big)$$

Quality factor

- Defined as $Q = \omega_0 / \gamma$

- This function defines the rate of energy loss due to the damping

- Over one oscillation period

$$E = E_0 \exp(-\gamma t)$$

$$\Rightarrow \dot{E} = -\gamma E$$

$$\Delta E = \dot{E}\Delta t = -\gamma E \Delta t$$

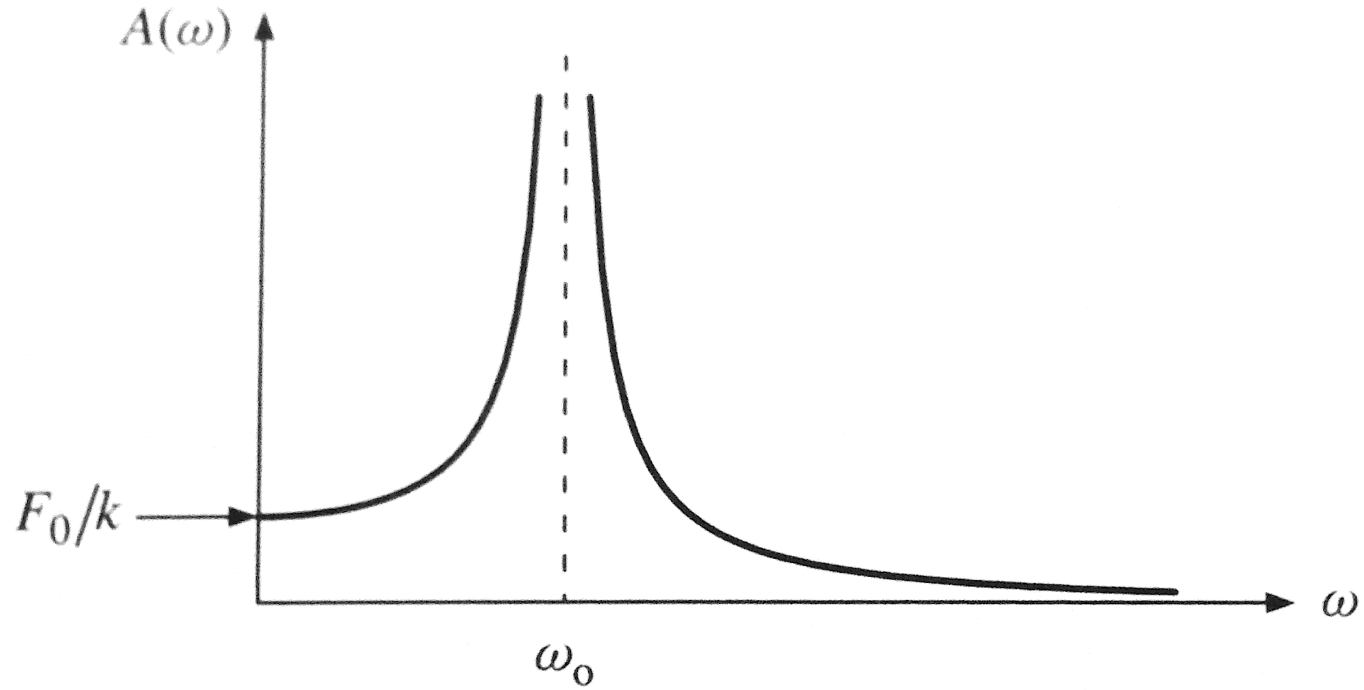
$$\frac{\Delta E}{E} = -\gamma\Delta t = -\gamma \frac{2\pi}{\omega} = -\frac{2\pi}{Q}$$

Forced oscillations, no damping

- Considered first the case with no damping, and with an applied oscillatory force
- We considered solutions where the variable responds with an oscillation of the same angular frequency as the applied force
- We allow the amplitude and phase factor to depend on the angular frequency of the applied force

$$x = A(\omega)\cos(\omega t - \delta)$$

Forced oscillations, no damping



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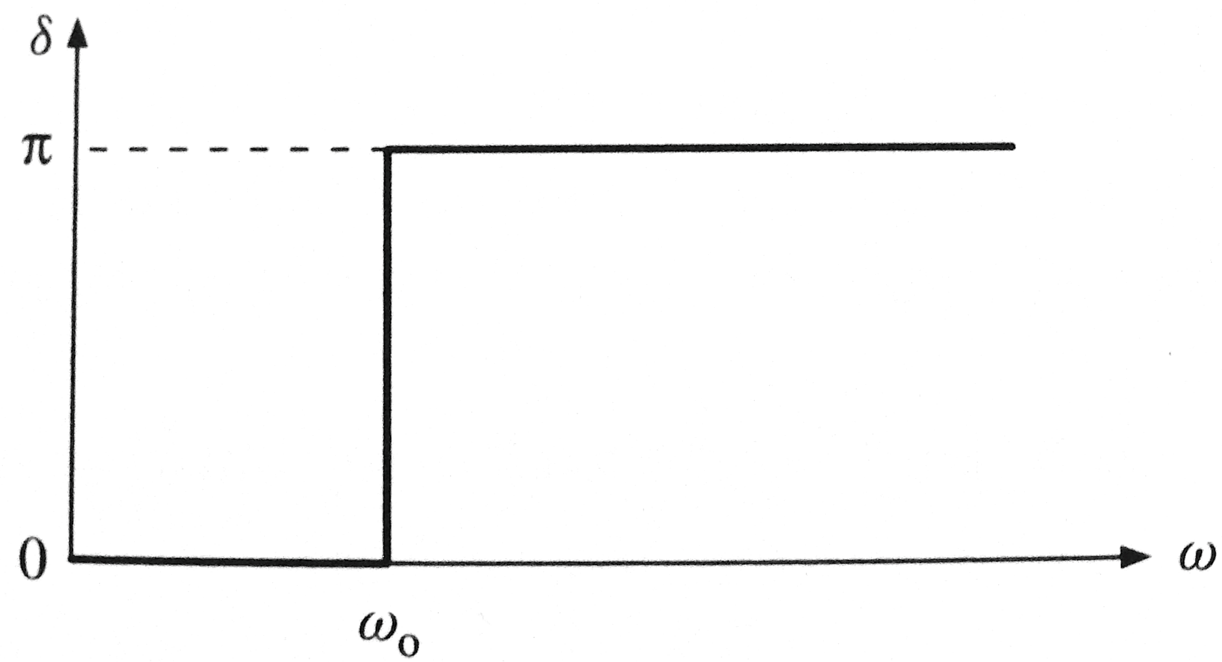
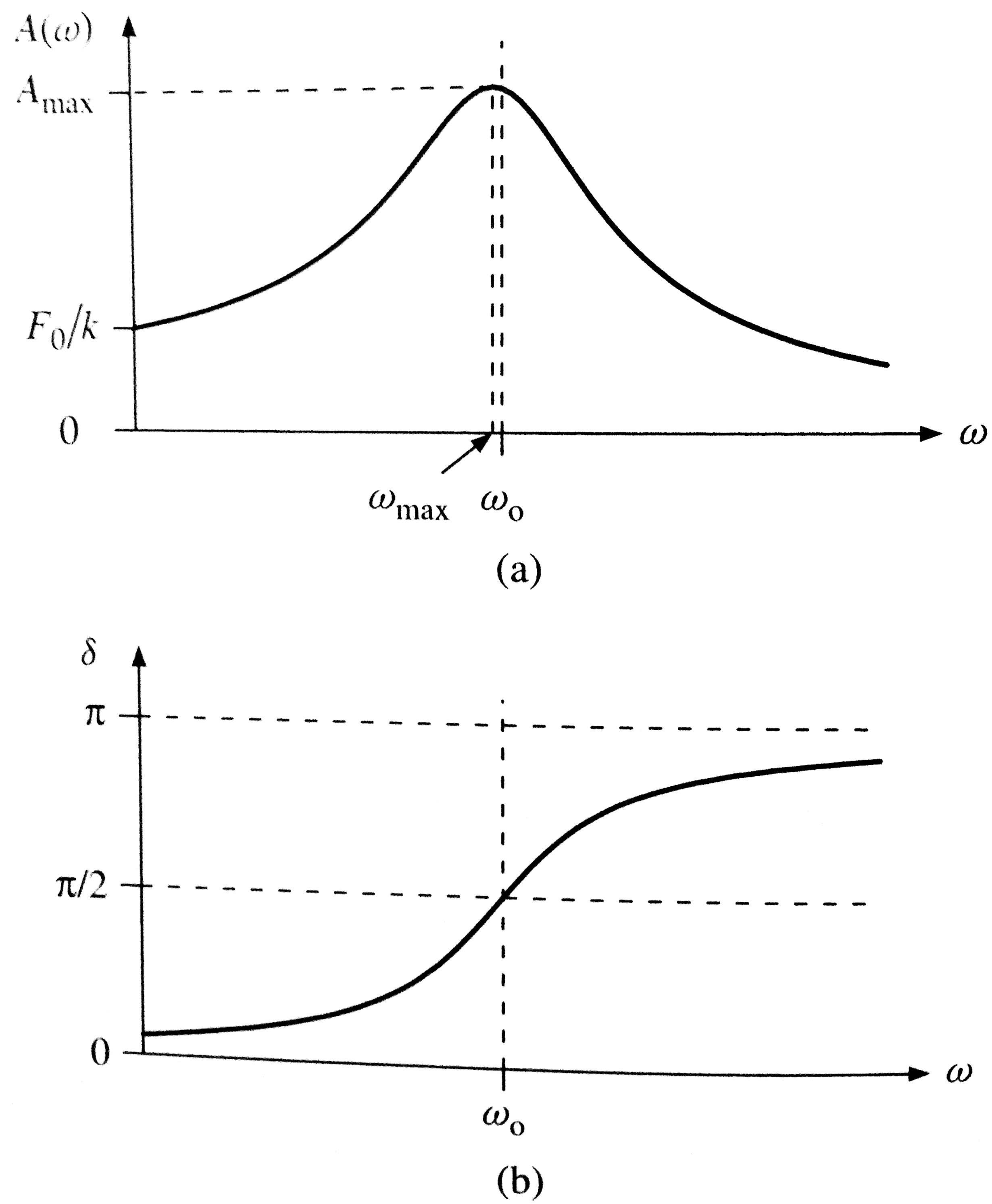


Forced oscillations with damping



Role of Q in forced oscillations with damping

- Q defines how narrow the resonance is; higher Q means sharper resonance
- Q defines how close the resonant frequency is to the natural frequency
- Q defines how quickly the change in phase occurs around the resonance point

$$A_{\max} = \frac{\omega_0 a / \gamma}{\left(1 - \gamma^2 / 4\omega_0^2\right)^{1/2}} = \frac{aQ}{\left(1 - 1/4Q^2\right)^{1/2}}$$

$$\omega_{\max} = \left(\omega_0^2 - \gamma^2/2\right)^{1/2} = \omega_0\left(1 - 1/2Q^2\right)^{1/2}$$

Steady state vs transient behaviour

- The previous slides show the steady state behaviour, which doesn't change with time

- The equations, however, permit some time dependence

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \omega_0^2 a \exp(i\omega t)$$
$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

- We can add the second equation to the first because it has a zero on the right-hand side

Steady state vs transient behaviour solutions

- The general solutions are

$$\ddot{x}+\gamma\dot{x}+\omega_0^2 x=\omega_0^2 a\exp(i\omega t)\Rightarrow x=A(\omega)\exp\big(i(\omega t-\delta)\big)$$

$$\ddot{x}+\gamma\dot{x}+\omega_0^2 x=0\Rightarrow x=A_0\exp(-\gamma t/2)\exp\big(i(\omega' t+\alpha)\big)$$

$$\omega'^2=\omega_0^2-\gamma^2/4$$

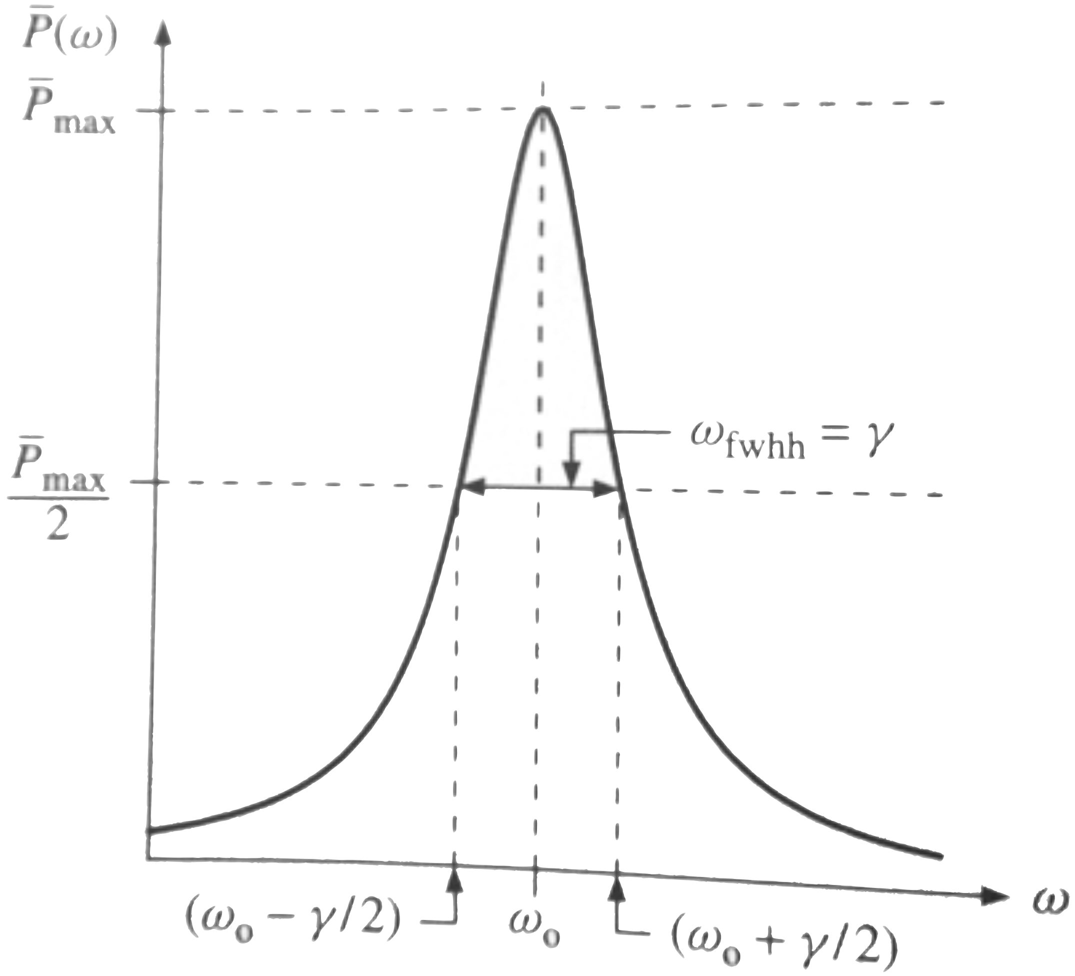
- These lead to an overall solution

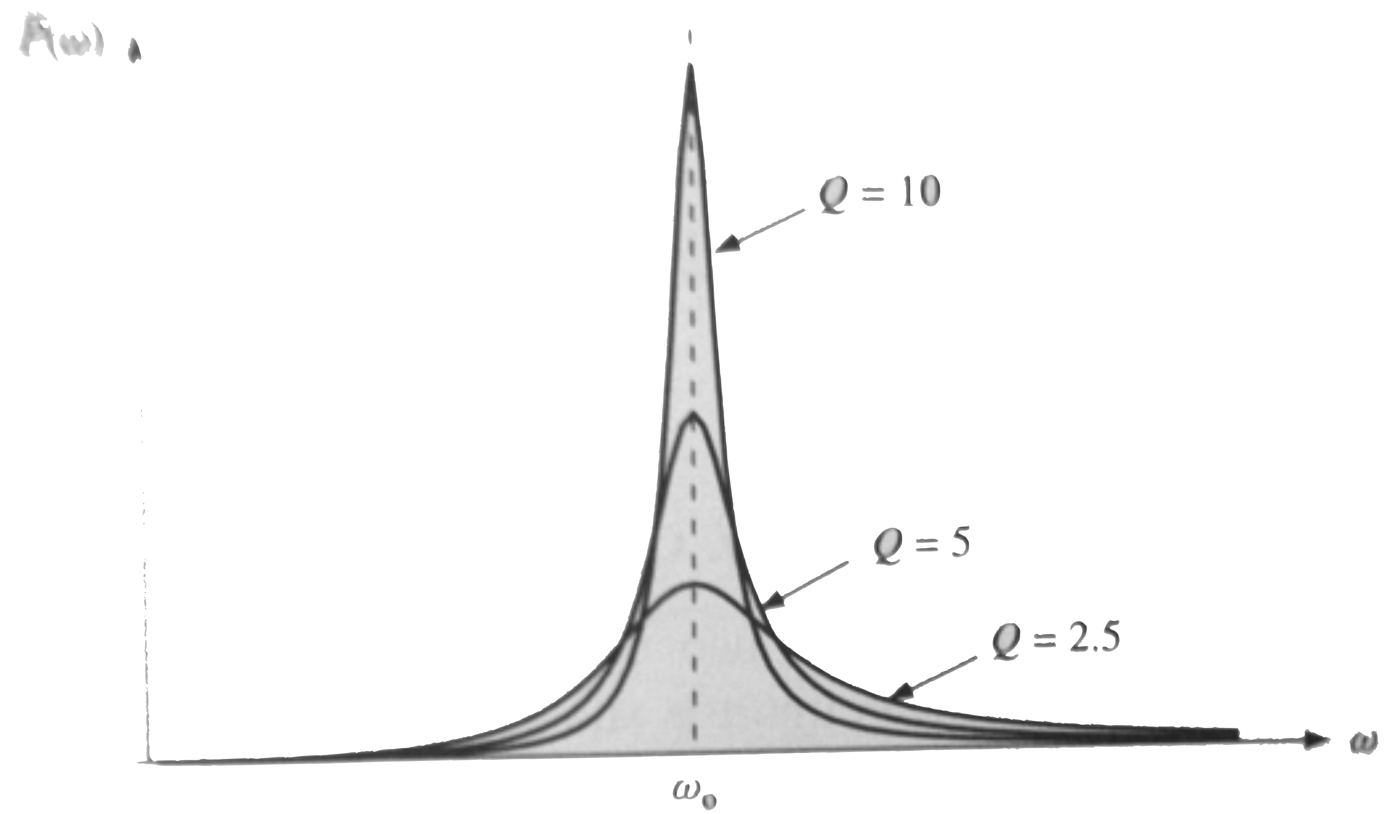
$$x=A(\omega)\exp\big(i(\omega t-\delta)\big)+A_0\exp(-\gamma t/2)\exp\big(i(\omega' t+\alpha)\big)$$

General features

- We get competition between the transient and steady state solutions – beating
- The transient state decays over time, leaving the steady state solution
- Important role of initial conditions

Power absorbed in a forced oscillator





Key concepts to define

- Resonance
- Phase change
- Transient state
- Beating

Coupled oscillators

- Key approach is to define the energy in terms of all the variables (more than one)
- Go from energy to forces, making sure you get the signs right!
- For each variable you will have both a force from the energy and a mass x acceleration
- I recommend that you try to write the resultant equations as matrix equations

Normal modes

- Fundamental motions of the coupled system
- Each normal mode has a single angular frequency
- All variables have a definite phase relation that is independent of amplitude
- Different normal modes of any system do not interact or exchange energy

Normal modes

- Fundamental motions of the coupled system
- Each normal mode has a single angular frequency
- All variables have a definite phase relation within a seingle normal mode, that is independent of amplitude
- Actual amplitude of each normal mode is set by the initial conditions
- Different normal modes of any system do not interact or exchange energy

Matrix equations, example of coupled pendulums

- Going from energy to forces

$$E = \frac{mg}{2L} x_a^2 + \frac{mg}{2L} x_b^2 + \frac{1}{2} k \left(x_a - x_b \right)^2$$

$$F_a = -\frac{\mathrm{d}E}{\mathrm{d}x_a} = -\frac{mg}{L} x_a - k \left(x_a - x_b \right) = m\ddot{x}_a$$

$$F_b = -\frac{\mathrm{d}E}{\mathrm{d}x_b} = -\frac{mg}{L} x_b + k \left(x_a - x_b \right) = m\ddot{x}_b$$

$$\ddot{x}_a + \omega_0^2 x_a + \omega_S^2 \left(x_a - x_b \right) = 0$$

$$\ddot{x}_b + \omega_0^2 x_b - \omega_S^2 \left(x_a - x_b \right) = 0$$

Matrix equations, example of coupled pendulums

- Assuming general solutions

$$x_a = C_a \cos\omega t \quad x_a = C_b \cos\omega t$$

$$-\omega^2 C_a + \omega_0^2 C_a + \omega_S^2 \left(C_a - C_b\right) = -\omega^2 C_a + \left(\omega_0^2 + \omega_S^2\right) C_a - \omega_S^2 C_b = 0$$

$$-\omega^2 C_b + \omega_0^2 C_b - \omega_S^2 \left(C_a - C_b\right) = -\omega^2 C_b + \left(\omega_0^2 + \omega_S^2\right) C_b - \omega_S^2 C_a = 0$$

$$\begin{pmatrix} \left(\omega_0^2 + \omega_S^2\right) - \omega^2 & -\omega_S^2 \\ -\omega_S^2 & \left(\omega_0^2 + \omega_S^2\right) - \omega^2 \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = 0$$

Matrix equations, example of coupled pendulums

- Solutions of matrix equation, angular frequencies as eigenvalues

$$\begin{vmatrix} \left(\omega_0^2+\omega_S^2\right)-\omega^2 & -\omega_S^2 \\ -\omega_S^2 & \left(\omega_0^2+\omega_S^2\right)-\omega^2 \end{vmatrix} = 0$$

$$\left(\left(\omega_0^2+\omega_S^2\right)-\omega^2\right)^2 = \left(\omega_S^2\right)^2$$

$$\omega^2 = \omega_0^2+\omega_S^2 \mp \omega_S^2$$

$$\omega_1^2 = \omega_0^2$$

$$\omega_2^2 = \omega_0^2 + 2\omega_S^2$$

Matrix equations, example of coupled pendulums

- Solutions of matrix equation, relative motions as eigenvectors obtained by substitution of each eigenvalue

$$\begin{pmatrix} \left(\omega_0^2+\omega_S^2\right)-\omega_0^2 & -\omega_S^2 \\ -\omega_S^2 & \left(\omega_0^2+\omega_S^2\right)-\omega_0^2 \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = \begin{pmatrix} \omega_S^2 & -\omega_S^2 \\ -\omega_S^2 & \omega_S^2 \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = 0 \Rightarrow C_{1a} = C_{1b}$$

$$\begin{pmatrix} \left(\omega_0^2+\omega_S^2\right)-\left(\omega_0^2+2\omega_S^2\right) & -\omega_S^2 \\ -\omega_S^2 & \left(\omega_0^2+\omega_S^2\right)-\left(\omega_0^2+2\omega_S^2\right) \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = \begin{pmatrix} -\omega_S^2 & -\omega_S^2 \\ -\omega_S^2 & -\omega_S^2 \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = 0 \Rightarrow C_{1a} = -C_{1b}$$

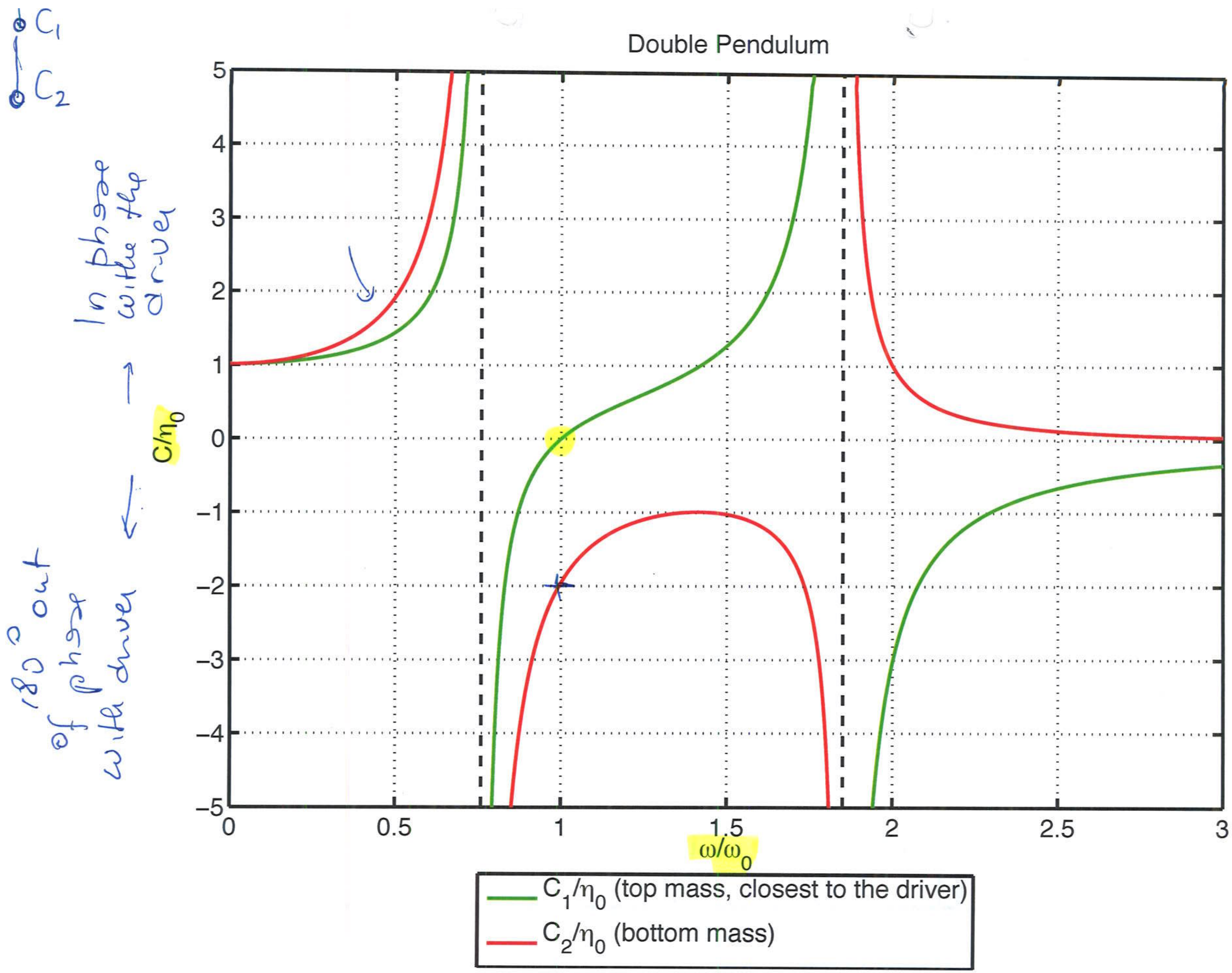
Examples tackled

- Two pendulums coupled with a spring
- CO_2 molecule (one solution was the trivial uniform displacement of zero frequency)
- Two weights on a single string: double pendulum

Driven coupled pendulum

- We considered the case where the applied force appears as an imposed sinusoidal displacement
- We now solved for the relative displacements of both weights as a function of applied angular frequency
- The resultant graph shows resonances at the angular frequencies of the two normal modes

Driven coupled pendulum: displacements



Driven coupled systems

- Resonance now occurs around the angular frequencies of the normal modes
- Example of absorption of infrared radiation by CO_2 molecule by the asymmetric stretch mode

Many coupled oscillators

- The number of normal modes = the number of variables
- Again, write the energy equation, noting that often each term may only involve a few variables (eg nearest neighbours)

Many coupled oscillators representing a crystal

- We introduced the travelling wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Here the coefficient on the right side represents the square of the velocity of the wave

- The point is that there are no boundaries that constrain the amplitudes at any point

Many coupled oscillators representing a crystal

- General solution

$$u(x,t) = u_0 \exp\left(i(kx - \omega t)\right)$$

$$\frac{\partial u}{\partial t} = -i\omega u_0 \exp\left(i(kx - \omega t)\right) \quad \frac{\partial^2 u}{\partial t^2} = -\omega^2 u_0 \exp\left(i(kx - \omega t)\right)$$

$$\frac{\partial u}{\partial x} = iku_0 \exp\left(i(kx - \omega t)\right) \quad \frac{\partial^2 u}{\partial x^2} = -k^2 u_0 \exp\left(i(kx - \omega t)\right)$$

$$\Rightarrow \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{k^2}\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{\omega^2}{k^2}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

New concepts

- Wave vector k, reflecting wavelength
- Wave equation now contains both angular frequency and wave vector
- Wave is sinusoidal in both time and space
- The maxima in the wave are now able to move with a velocity given as c

Group and phase velocity

- Phase velocity is the velocity of the motion of the maxima

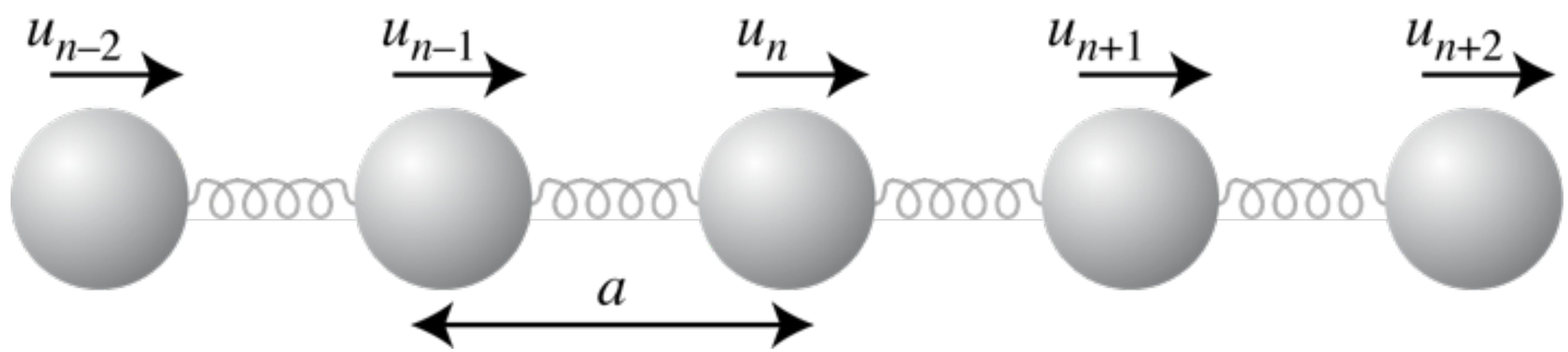
$$c_{\mathrm{p}} = \frac{\omega}{k}$$

- Group velocity represents the velocity of a wave packet, frequently representing the velocity of the energy flow

$$c_{\mathrm{g}} = \frac{\partial\omega}{\partial k}$$

- A standing wave has a group velocity of zero

Normal modes of a crystal



- We obtained force equations for the displacement of any atom
- We used the travelling wave equation to represent the displacement of the same atom
- The travelling wave equation gives the acceleration, which we equate to the force to enable us to derive and equation for the angular frequency of the wave

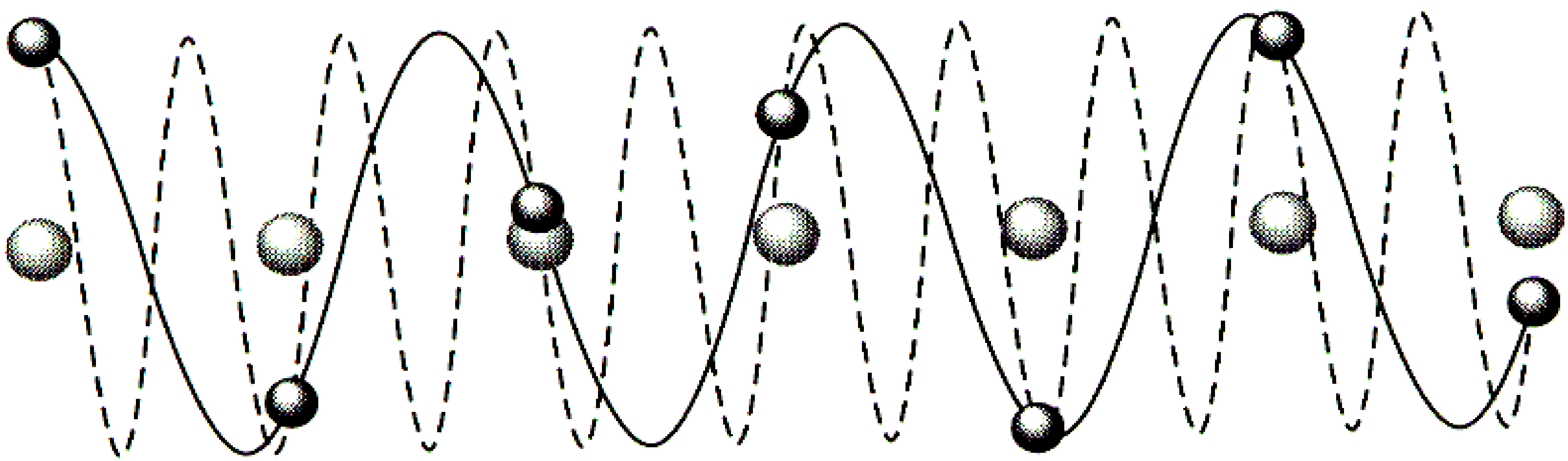
Solution

$$\omega = \sqrt{\frac{4J}{m}}\left|\sin(ka/2)\right|$$

- Periodic solution – we showed that this is a consequence of the fact that there are groups of values of k that give rise to identical atom displacements
- In the limit of small k we have a linear relationship between angular frequency and wave vector equivalent to a sound wave

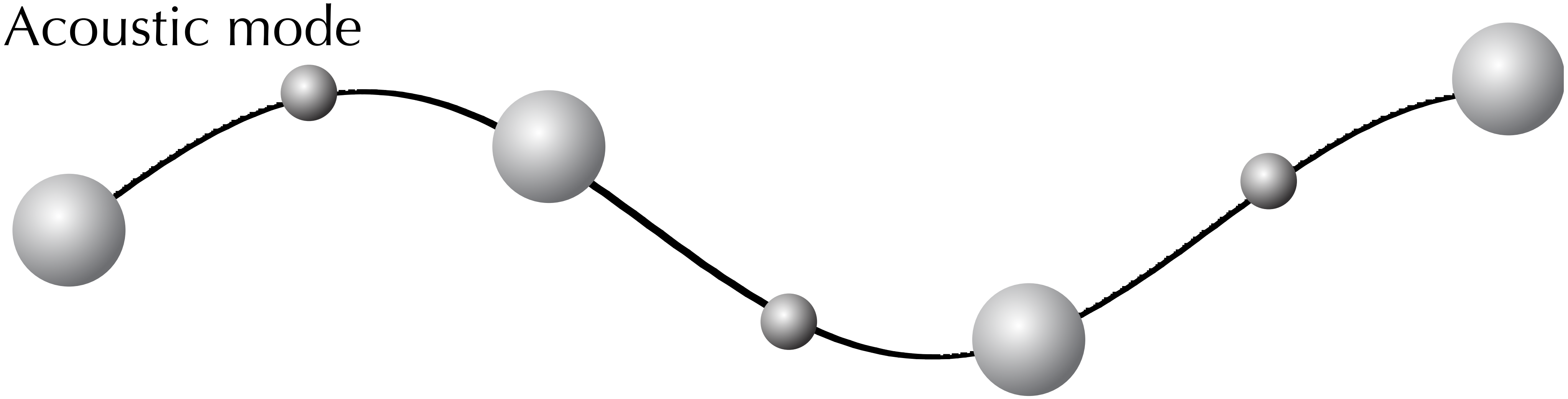
Representation of the periodic solution

$$\omega = \sqrt{\frac{4J}{m}} \left| \sin(ka/2) \right|$$

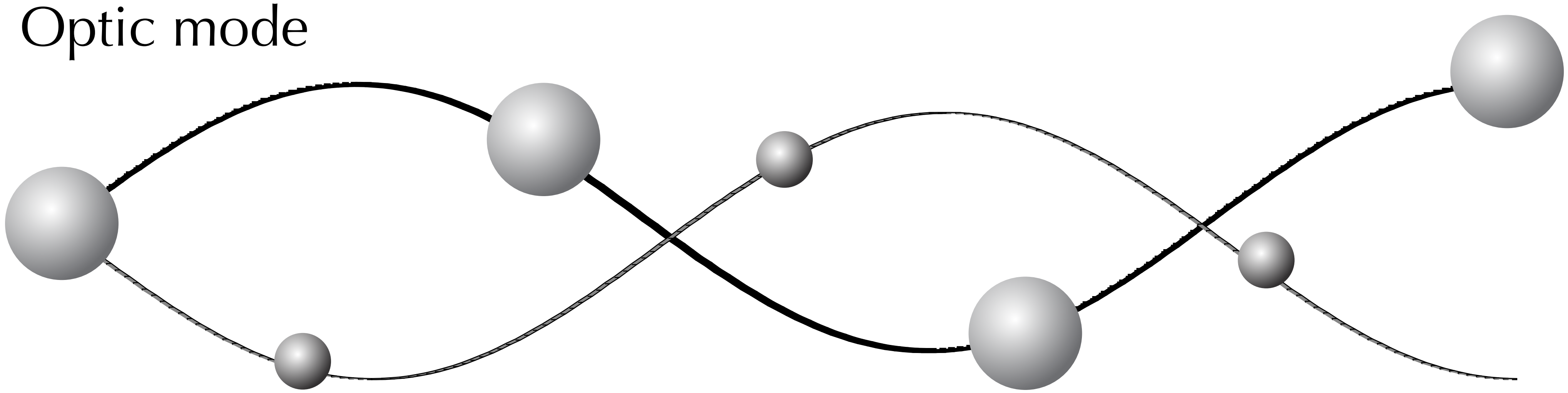


More complex crystals

Acoustic mode



Optic mode



Later topics

- We looked at macroscopic realisations of travelling waves
- We also looked at transmission and reflection from interfaces between media

Some summary points 1

- We need for any example to establish the forces acting. The easiest thing is to set up an equation for the potential energy – whether it be for springs or gravitational potential energy – and compute the force as the negative of the derivative. Provided you remember the sign of the derivative, you will always get the signs of the forces correctly. Getting a consistent set of signs is the biggest challenge.
- Note that this works equally well for a complex set of coupled oscillators as it does for a single oscillation.

Some summary points 2

- For damping, there is an additional force that is proportional to the velocity, with a minus sign because it acts against the direction of motion. The point of this force is that it doesn't act when the object is at rest.
- You need to then equate the force on the object to mass times acceleration.
- For forced systems, you need to be add in the external force. Note that sometimes this is an explicit force, but often it is given in the form of an enforced periodic displacement. We have dealt with both systems.

Some summary points 3

- Often we have to add up waves, using standard trigonometric relations. Recall that when you add two waves of similar frequency, you get a product of two terms, one being the average frequency and one being derived from the difference in frequencies. This leads to beating, which is important in transient phenomena.

- We have dealt with two types of wave equation. One where there is no spatial derivative (the fixed oscillators) and one where there is a second derivative with respect to distance. The latter gives travelling waves.

Some summary points 4

- We have seen three derivations of the differential equation for travelling waves, both atomic and continuum. You will also see a derivation in electromagnetic theory, where you can find electromagnetic waves with a velocity of light given by fundamental parameters.
- The solution is a wave where you get not only the frequency times time term but also a wave vector times position component of the argument of the sine wave.

Some summary points 5

- You need to understand the concept of wave vector. It is not hard, and is a common concept in Physics.
- Be aware that although we like to think in terms of real numbers, many problems of waves are best tackled using complex number representations. We saw that using complex numbers gave a much easier solution to the damped oscillator, where we got both under-damped and over-damped solutions from one equation, and where we didn't need to make any prior assumptions.
- For complex cases, you need to work with matrices!