BSc/MSci MidTerm Test

PHY-217: Vibrations and Waves

Time allowed: 1 hr 15 minutes

Date: 13th November 2012

Time: 9:15 – 10:30 am

Answer ALL questions in section A.

Answer ONLY ONE question from section B.

Section A carries 60 marks and Section B carries 40 marks.

An indicative marking-scheme is shown in square brackets [...] after each part of a question.

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiner: Prof Martin Dove

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SECTION A

Attempt answers to ALL questions.

A1

An object executing one-dimensional simple harmonic motion around an equilibrium position can have its displacement x at any time t described by the function

$$x(t) = x_0 \cos(\omega t + \phi)$$

i) Define the quantities x_0, ω and ϕ. [4 marks]

x_0 is the amplitude, defining the maximum value of the displacement, ω is the angular frequency equal to 2π divided by the period of oscillation, and ϕ is the phase angle defining where the maxima and minima are relative to the starting time.

ii) Sketch a graph of this function with appropriate labels on the axes. [2 marks]

The graph will be a sine curve with maximum and minimum values on the vertical axis of $+x_0$ and $-x_0$ respectively. The graph should show the period equation to 2π divided by the angular frequency. The value of the function at zero time will be $\cos\phi$. The positions of the maxima or minima will be offset from the value of $n\pi/\omega$ by ϕ; the definition of the function at zero time will define the sign of ϕ.

A2

Consider a simple pendulum consisting of an object of mass m hanging at the end of a thin string of negligible mass. The string has length L. Denote the displacement of the object when swinging as x and y in the horizontal and vertical directions respectively.

i) The pendulum is rotated by an angle θ whilst keeping the string taught. Write equations for x and y in terms of θ, and hence obtain an equation for y in terms of x. [2 marks]

$$y = \frac{1}{2} L\theta^2$$

$$x = L\sin\theta \simeq L\theta$$

$$y = \frac{1}{2L} x^2$$

ii) Noting that the gravitational energy associated with a displacement is equal to the product of the mass, gravitational constant $g = 9.8\ \mathrm{ms^{-2}}$, and the vertical displacement y, write the energy of the pendulum as a function of x. [2 marks]

$$E = mgy = \frac{mg}{2L} x^2$$

iii) By differentiating the potential energy, write an equation for the force experienced by the object for any rotation of the pendulum in terms of the horizontal displacement x. [2 marks]

$$F = -\frac{\mathrm{d}E}{\mathrm{d}x} = -\frac{mg}{L} x$$

iv) By equating the force with *mass* × *acceleration*, write the differential equation that describes the motion of the pendulum in terms of the horizontal displacement x. [2 marks]

$$m\ddot{x}+\frac{mg}{L}x=0 \quad \Rightarrow \quad \ddot{x}+\frac{g}{L}x=0$$

v) Obtain an equation for the angular frequency of oscillations of the pendulum. [2 marks]

$$x = A\cos\omega t$$

$$-\omega^2 A\cos\omega t+\frac{g}{L}A\cos\omega t=0$$

$$-\omega^2+\frac{g}{L}=0$$

$$\omega=\sqrt{\frac{g}{L}}$$

A3

Sketch a labelled graph showing *both* the potential and kinetic energy of an undamped oscillator as a function of time. [4 marks]

The graph should show two sine-squared curves (origin doesn't matter) that are exactly out of phase, so that the maximum of one function corresponds to the zero (minimum) of the other. The sum will be a constant.

How will this graph change with light damping?

Both energy functions will by multiplied by a decaying exponential function.

A4

The differential equation that describes a damped oscillator can be written as

$$\ddot{x}+\gamma\dot{x}+\omega_0^2 x=0$$

where x is the displacement from equilibrium.

Define the following terms using the parameters in the differential equation, and describe the motion in each case:

i) Underdamped oscillator [2 marks]

This is the case for $\omega_0^2>\gamma^2/4$. The motion is oscillatory but the amplitude of oscillation decays exponentially with time.

ii) Overdamped oscillator [2 marks]

This is the case for $\omega_0^2<\gamma^2/4$. The motion is not oscillatory, but instead is a slow decay from the initial displacement towards zero displacement at long times.

iii) Critically damped oscillator [2 marks]

This is the case for $\omega_0^2 = \gamma^2 / 4$. Like the overdamped case, the motion is not (or is barely at best) oscillatory, but instead there is a decay from the initial displacement towards zero displacement. This is rather faster than the overdamped case,

A5

Assume a solution for the differential equation given in question A4 of the form

$x = A_0 \exp(i\beta t)$

i) Substitute this solution into the differential equation to obtain an equation for β. [4 marks]

$x(t) = x_0 \exp(i\beta t)$

$\dot{x}(t) = x_0 i\beta \exp(i\beta t)$

$\ddot{x}(t) = -x_0\beta^2 \exp(i\beta t)$

$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$

$\Rightarrow -x_0\beta^2 \exp(i\beta t) + ix_0\gamma\beta \exp(i\beta t) + \omega_0^2 x_0 \exp(i\beta t) = 0$

$\Rightarrow -\beta^2 + i\gamma\beta + \omega_0^2 = 0$

$\Rightarrow \beta^2 - i\gamma\beta - \omega_0^2 = 0$

$\Rightarrow \beta = \dfrac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2} = \dfrac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2 / 4}$

ii) Use the solutions for β to obtain equations for x in the underdamped and overdamped cases. [4 marks]

For the underdamped case the square root is a real quantity, and we have the motion

$$x(t) = x_0 \exp(i\beta t) = x_0 \exp(-\gamma t / 2)\exp\left(i\sqrt{\omega_0^2 - \gamma^2 / 4t}\right) = x_0 \exp(-\gamma t / 2)\exp\left(i\omega t\right)$$

For the case overdamped case, the square root is an imaginary quantity, and we have the motion

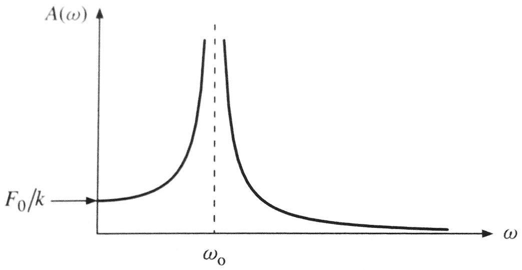
$$x(t) = x_0 \exp(i\beta t) = x_0 \exp(-\gamma t / 2)\exp\left(\pm\sqrt{\gamma^2 / 4 - \omega_0^2 t}\right)$$

The solution involves taking both positive and negative solutions.

A6

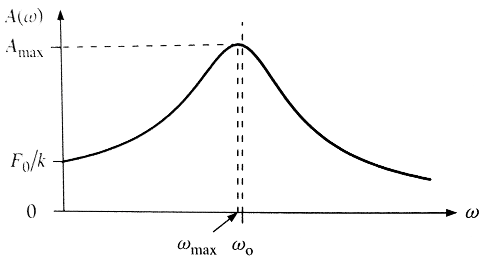
Sketch and label plots of the amplitudes of a forced oscillator in the cases of i) no damping, and ii) light damping. Explain in words the key differences. [6 marks]

The graph for no damping should look like



The important points to note are that there is divergence at the resonant frequency, that the function falls to zero at large frequency, and to a constant value at zero frequency

For the damped case, this is modified to



The main differences to not are that the function doesn't go to infinity, and that the peak position is a little lower than the resonant frequency.

A7

i) Define the quality factor Q for a damped oscillator. [4 marks]

Formally Q is defined as

$$Q = \frac{\omega_0}{\gamma}$$

In terms of energy loss per period of oscillation, this is determined by the value of Q:

$$\Delta t = \frac{2\pi}{\omega}$$

$$\frac{\Delta E}{E} = -\gamma \frac{2\pi}{\omega} = -\frac{2\pi}{Q}$$

ii) How does the shape of a graph of absorbed energy for a forced oscillator depend on Q? [4 marks]

It is a resonance peaking at the resonant frequency, with a width that is proportional to Q.

A8

i) Define the term "beating" for an oscillator. Give *one* example of how beating can be created. [4 marks]

Beating arises from the summation of two oscillations with similar frequency. The result of adding these two sinusoidal curves is to form a product of a slow vibration (given by the difference in frequencies) and a slow vibration (given by half the sum of the two vibrations). So you would hear the fast vibration but it’s amplitude would go to zero at times determined by the slow vibration.

You need two oscillations. These could be two separate sources, such as two tuning forks that have a slightly different frequency, or later in the lectures we saw that coupled oscillators can show beating. In the lectures on force oscillations we saw beating emerge as transient behaviour in the forced damped oscillator.

i) Describe the role of beating in the form of the transients established at the onset of an applied sinusoidal force to a lightly damped oscillator. [6 marks]

In the case of a forced damped oscillator, there are two solutions, namely the solution of the free oscillator that will decay after a certain period of time, and the steady state solution. The former will oscillate at the natural frequency, and the second at the frequency of the forcing motion. This means there are two oscillation frequencies, and before the natural oscillations have decayed the interaction of two separate oscillations will produce beating.

SECTION B

Answer only ONE of the two questions in this section. No credit will be given for answers to a second question.

B1

Consider a molecule consisting of two atoms of mass m_1 and m_2, and a bond force constant k.

i) Denote the displacements of each atom as u_1 and u_2 respectively, and write an equation for the potential energy of the bond in terms of the displacements of both atoms. [6 marks]

$$E = \frac{1}{2}k\left(u_1 - u_2\right)^2$$

ii) Derive equations for the force experienced by each atom. [6 marks]

$$F_1 = -\frac{\mathrm{d}E}{\mathrm{d}u_1} = -k\left(u_1 - u_2\right)$$

$$F_2 = -\frac{\mathrm{d}E}{\mathrm{d}u_2} = +k\left(u_1 - u_2\right)$$

iii) Assume the motions of both atoms move follow a sinusoidal vibration of angular frequency ω, but with each atom experiencing a different amplitude. Write a differential equation for the motion of each atom, and substitute in the corresponding sinusoidal function. Derive equations for the angular frequency and the relative displacements of the two atoms. [8 marks]

$$u_1 = a_1 \cos\omega t \quad ; \quad u_2 = a_2 \cos\omega t$$

$$\ddot{u}_1 = -\omega^2 a_1 \cos\omega t \quad ; \quad \ddot{u}_2 = -\omega^2 a_2 \cos\omega t$$

$$F_1 = -k\left(a_1 - a_2\right)\cos\omega t = m_1\ddot{u}_1 = -m_1\omega^2 a_1 \cos\omega t$$

$$\Rightarrow -\omega^2 a_1 + \frac{k}{m_1}\left(a_1 - a_2\right) = 0$$

$$F_2 = +k\left(a_1 - a_2\right)\cos\omega t = m_2\ddot{u}_2 = -m_2\omega^2 a_2 \cos\omega t$$

$$\Rightarrow -\omega^2 a_2 - \frac{k}{m_2}\left(a_1 - a_2\right) = 0$$

$$\Rightarrow \begin{pmatrix} \frac{k}{m_1} - \omega^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{k}{m_2} - \omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

The solution is that the determinant of the matrix is zero

$$\begin{vmatrix} \frac{k}{m_1}-\omega^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{k}{m_2}-\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{k}{m_1}-\omega^2\right)\left(\frac{k}{m_2}-\omega^2\right)-\frac{k^2}{m_1 m_2} = \omega^4 - \omega^2\left(\frac{k}{m_1}+\frac{k}{m_2}\right) = 0$$

There are two solutions. The trivial one is that the frequency is zero. The one we want is obtained as

$$\omega^4 - \omega^2\left(\frac{k}{m_1}+\frac{k}{m_2}\right) = 0$$

$$\Rightarrow \omega^2 - \left(\frac{k}{m_1}+\frac{k}{m_2}\right) = 0$$

$$\Rightarrow \omega^2 = \left(\frac{k}{m_1}+\frac{k}{m_2}\right) = k\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$$

We have the following data for the carbon monoxide molecule:

Bond energy = –1077 kJ/mol

Bond length = 1.13×10^{-10} m

Vibration frequency = 64 THz

The bond can be modelled by a Morse potential energy function of the form

$$E(r) = E_0\left[\exp 2\alpha(r_0 - r) - 2\exp\alpha(r_0 - r)\right]$$

where r is the distance between the two atoms, and the other quantities are parameters in the model.

i) Show by differentiation that the equilibrium separation is at $r = r_0$. [6 marks]

$$\frac{\mathrm{d}E}{\mathrm{d}r} = E_0\left[-2\alpha\exp 2\alpha(r_0 - r) + 2\alpha\exp\alpha(r_0 - r)\right] = 0 \text{ if } r - r_0$$

ii) Obtain by second differentiation an equation for k in terms of α and E_0. [8 marks]

$$\frac{\mathrm{d}^2E}{\mathrm{d}r^2} = E_0\left[4\alpha^2\exp 2\alpha(r_0 - r) - 2\alpha^2\exp\alpha(r_0 - r)\right]$$

$$\left.\frac{\mathrm{d}^2E}{\mathrm{d}r^2}\right|_{r=r_0} = 2\alpha^2 E_0$$

iii) From the above data, obtain values for the parameters α, r_0 and E_0. [6 marks]

$r_0 = 1.13 \times 10^{-10}$ m

E_0 = 1077 kJ/mol (watch for the sign!)

$$\omega^2 = 4\pi^2 f^2 = 2\alpha^2 E_0\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$$

$$\alpha^2 = \frac{2\pi^2 f^2}{E_0}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^{-1}$$

$$= \frac{2\pi^2\left(64\times 10^{12}\right)^2}{1077\times 10^3}\times\left(\frac{1}{12\times 10^{-3}}+\frac{1}{16\times 10^{-3}}\right)^{-1}$$

$$= 1.09\times 10^{19}\ \mathrm{m}^{-2}$$

$$\alpha = 0.33\times 10^{10}\ \mathrm{m}$$

The trick with this is to get the units right, including factors of 2π. So long as the students are on the right track, credit should be given even if the units go wrong, but reserving one mark for correct use of units.

B2

Four people of total mass 300 kg get into a car. The car springs are compressed and the body of the car is lowered by 3 cm. Whilst driving, the car hits a bump in the road and oscillates vertically with a period of 0.75 s.

i) Derive the value of the spring constant for the car suspension (the acceleration due to gravity can be taken as $g = 9.8\ \mathrm{ms}^{-2}$). [10 marks]

Force applied is mass time g = spring constant multiplied by the displacement. This leads to

$$k = \frac{mg}{d} = \frac{300\times 9.8}{3\times 10^{-2}} = 9.8\times 10^4\ \mathrm{N/m}$$

ii) Derive the total mass of the car and occupants, and hence of the car, assuming that damping is negligible. [10 marks]

$$\omega^2 = (2\pi/T)^2 = k/m$$

$$m = k/\omega^2 = kT^2/(2\pi)^2 == 9.8\times 10^4\times 0.75^2/(2\pi)^2 = 1396\ \mathrm{kg}$$

$$m_{\mathrm{car}} = 1396 - 300 = 1096\ \mathrm{kg}$$

iii) The damping on the suspension allows the oscillations to have decayed after 4 seconds. Assuming damping occurs through the existence of a force that is proportional to $-bv$, where b is a constant and v is the velocity, obtain a value for b and hence for the quality factor Q. [20 marks]

This is a deliberately ambiguous because I was not after people remembering exact details. But I am expecting students to assume that the 4 s time is the decay time, when for the damped oscillator the damping is equal to $\exp(-\gamma t/2)$, we might identify $\gamma/2$ as $1/4\ \mathrm{s}^{-1}$. From the theory of the damped oscillator, $\gamma = b/m$. Thus we have $b = 2\times(1/4)\times 1396 = 698\ \mathrm{N.s.m}^{-1}$. Variants on this that try to more complex interpretation of the decay time would be fine. Q is given as $\omega_0/\gamma = (2\pi/0.75)/(1/2) = 16.8$.

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