Topic 9: Macroscopic systems

Recap

1. Early stages we looked at single oscillations, which would involve oscillation about a mean or equilibrium point. Time dependence by no spatial dependence.
2. We introduced first the concept of damping, namely energy loss through the vibration.
3. We introduced second the concept of forcing, where you feed in energy. Key result was that energy absorption increases closed to the natural resonant frequencies. We combined damping with forcing in order to avoid infinity amplitudes that we know do not exist, although experience tells us that there are cases where hitting resonance gives huge amplitudes. The concepts of damping and forcing apply to all types of vibration, but going forward we do not do damping again, and we only do forcing for simpler systems. But the principles work going forward.
4. The next stage was coupled oscillators with some tethering, so again we got time dependence but no spatial dependence. We looked again at forcing, and found some interesting dependence of amplitudes on frequency.
5. We understood the concept of normal modes, where each mode represents a specific phase relationship between displacements. Normal modes are the natural vibrations. One key point is that if we represent the displacements in terms of normal modes, and then compute the potential and kinetic energy, we find that the normal modes represent the combination of displacements such that the energy written in terms of normal modes has no term corresponding to two normal mode amplitudes. We call this orthogonality.
6. We then looked at multiple coupled systems. We looked in detail at chains of balls with both ends connected, and we guessed that the normal modes are the sinusoidal functions with nodes at each ends. The number of normal modes is equal to the number of objects, and we add modes with wavelengths = length/*n*, where *n* is the order and with all values between 1 and the number of objects. These modes are called standing waves because they don't move.
7. Last time we removed the ends of the medium. This allowed us to set up a number of new concepts.
8. Concept 1: the wave equation with spatial extent:

$$\frac{\partial^2 u}{\partial t^2} = \frac{K}{\rho}\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

9. Solutions were the travelling wave solutions

$$u(x,t) = u_0 \exp\big(i(kx - \omega t)\big)$$

10. We have the same angular frequency to describe the time dependence, but we now have the new quantity called the wave vector, whose value is 2π/wavelength. Wavelength is to space what the period is to time, namely the distance or time over which the wave repeats.

11. In three dimensions, the wave vector is normal to the wave front, and gives the direction in which the wave travels, in addition to providing the information on the wavelength.

12. The form of the above solution means that the maxima move forward in space with a velocity equal to $v_g = \omega / k$, which is called the "phase velocity". It is defined in this way.

    If you have a train of waves, the phase velocity gives the velocity at which the which the front moves forward.

13. We analysed a simple model – chain of atoms with free ends – and found that travelling waves provide a good solution. We obtained the angular frequency for any wave vector, and found that the angular frequency was only strictly proportional to wave vector in the limit of small wave vector. At higher wave vector we found that he angular frequency started to dip.

14. The case where frequency is not proportional to wave vector is called "dispersion".

15. The graph of angular frequency vs wave vector is called the "dispersion curve".

16. At the limit of wavelength = twice repeat, we found that the graph of angular frequency was a maximum, with the slope = 0 (draw dispersion curve for reminder). Note that in a crystal the wave length spans the range from the size of the crystal down to twice an atomic bond length.

17. The differential $\partial\omega / \partial k = 0$ at this point.

18. This differential is called the "group velocity". It is the velocity at which a wave envelope moves. It is often usually the velocity at which the energy flows.

19. Consider a localised distribution. This is made up of many waves that all have a maximum at the position of the disturbance (draw picture and highlight the idea that the localised disturbance can be defined as a superposition of waves via the Fourier transform).

20. If you have a chain with two types of atoms, you get similar travelling wave solutions, but now in-phase and out-of-phase solutions; acoustic and optic modes. The optic modes will not have zero frequency at zero wave vector.

21. In a crystal you get all modes excited. The vibrations move the atoms from their equilibrium positions, so that the Amplitude is proportional to temperature, and to the inverse of the square of the frequency. Because of the dispersion, the pattern of atomic motions never repeats. The motion looks chaotic but in reality is fully deterministic and we now understand the solutions.

Example of wave-particle duality

Consider a particle of mass m. It's momentum and kinetic energy are

$$p = mv \quad ; \quad E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

From wave-particle duality we can write the momentum as

$$p = \frac{h}{\lambda} = \frac{h}{2\pi} \times \frac{2\pi}{\lambda} = \hbar k$$

It follows that the energy can be written as

$$E = \frac{\hbar^2 k^2}{2m} \equiv hf = \hbar\omega$$

In this case the phase velocity is not very interesting. The group velocity though is obtained as

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{\hbar}\frac{\hbar^2 k}{m} = \frac{p}{m} = v$$

So in wave particle duality, the group velocity gives the velocity of the particle. Group velocity is often associated with the velocity of the energy flow, which is going to be the velocity of the particle.

Waves in macroscopic systems

1. Longitudinal vibrations of a rod

Consider a small slice at position x and thickness Δx. This displaces by a distance u, but also stretches by an amount Δu.

So the average strain is $\Delta u / \Delta x$

And the average stress is $Y \Delta u / \Delta x$

So move along one slice, and you have have

stress at position $x + \Delta x$ = $Y \Delta u / \Delta x + \frac{\partial}{\partial x}\left(Y \Delta u / \Delta x\right)$

Cross sectional area is A. We can thus express the force at either end of a slice in the infinitesimal limit as

$$F_1 = AY\frac{\partial u}{\partial x}$$

$$F_2 = AY\frac{\partial u}{\partial x} + AY\frac{\partial^2 u}{\partial x^2}\Delta x$$

$$F_2 - F_1 = AY\frac{\partial^2 u}{\partial x^2}\Delta x$$

Now we have to think about the acceleration of the matter in the slab. Density is ρ, so mass is $\rho \Delta x A$. Thus we have

$$AY\frac{\partial^2 u}{\partial x^2}\Delta x = \rho A \Delta x \frac{\partial^2 u}{\partial t^2}$$
$$\Rightarrow \frac{Y}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

With the velocity of sound given as the square root of Y/ρ. No surprises since this is not so different from the chain model.

Example of aluminium

Y = 6 x 10^{10} GPa

ρ = 2700 kg/m^2

v = 4714 m/s

Note that the same analysis will actually hold also for a fluid medium for longitudinal motions.

But if we have a bar fixed at both ends, perhaps we don't need the travelling waves after all, but instead need a standing wave.

Bar has length L and from before we note that it will have standing waves of the form

$$u(x,t) = u_0 \exp\big(i(kx - \omega t)\big) + u_0 \exp\big(i(kx + \omega t)\big)$$
$$k = \frac{2\pi}{\lambda} = \frac{2\pi n}{2L} = \frac{\pi n}{L}$$
$$u(L,t) = u_0\big(\exp(-i\omega t) + \exp(+i\omega t)\big) = 2\cos\omega t = u(0,t)$$

Let's add two waves in opposite directions

$$u(x,t) = u_0 \exp\big(i(kx - \omega t)\big) + u_0 \exp\big(i(kx + \omega t)\big)$$
$$k = \frac{2\pi}{\lambda} = \frac{2\pi n}{2L} = \frac{\pi n}{L}$$
$$u(L,t) = u_0 \exp(ikx)\big(\exp(-\omega t) + \exp(+\omega t)\big) = 2\exp(ikx)\cos\omega t = u(0,t)$$

2. Transverse vibrations of a stretched string

Consider a string that is bent, such that a segment has angle θ with the normal at one end and θ + Δθ at the other. This will generate a difference in forces across the segment.

Suppose it is already under tension *T*. The forces in the x and y directions are going to be

$$F_x = T\cos(\theta + \Delta\theta) - T\cos\theta = 2T\sin(\theta + \Delta\theta/2)\sin(\theta - \Delta\theta/2) \approx T\theta^2 \sim 0$$
$$F_y = T\sin(\theta + \Delta\theta) - T\sin\theta \approx T\Delta\theta$$

We need to link this to a mass x acceleration:

$$T\Delta\theta = (\mu\Delta x)\ddot{y}$$

where μ is mass per unit length

We need to compute a value for $\Delta\theta$. We use the relationship

$$\Delta\theta = \frac{\partial\theta}{\partial x}\Delta x$$

$$\tan\theta = \frac{\partial y}{\partial x}$$

$$\frac{\partial \tan\theta}{\partial x} = \frac{\partial \tan\theta}{\partial\theta}\frac{\partial\theta}{\partial x} = \sec^2\theta\frac{\partial\theta}{\partial x} = \frac{\partial^2 y}{\partial x^2}$$

$$\sec^2\theta \simeq 1 \quad \Rightarrow \quad \frac{\partial\theta}{\partial x} \simeq \frac{\partial^2 y}{\partial x^2} \quad \Rightarrow \quad \Delta\theta = \frac{\partial^2 y}{\partial x^2}\Delta x$$

Hence we have

$$T\Delta\theta = T\frac{\partial^2 y}{\partial x^2}\Delta x = (\mu\Delta x)\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}$$

This is a standard wave equation with the velocity of sound given as

$$c = \sqrt{\frac{T}{\mu}}$$

Now let's think about where we have come from, and set this up with fixed ends, like a string on a musical instrument.

So let us assume a general solution of the form

$$y(x,t) = f(x)\cos\omega t$$

Using the cosine rather than the sine so that we start with a displacement.

We then have

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x)\cos\omega t$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}\cos\omega t$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 f}{\partial x^2} \Rightarrow -\omega^2 f(x)\cos\omega t = \frac{T}{\mu}\frac{\partial^2 f}{\partial x^2}\cos\omega t$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\omega^2}{c^2}f = 0$$

This is familiar territory. Since we need $f = 0$ at $x = 0$, our solution is

$$f(x) = A\sin\left(\frac{\omega x}{c}\right)$$

With a string fixed at both ends, we also need the condition that $f = 0$ at $x = L$. So we need to have

$$f(L) = A\sin\left(\frac{\omega L}{c}\right) = 0$$

and hence

$$\frac{\omega L}{c} = n\pi \quad \Rightarrow \quad \nu_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

For example, the E string of a violin is set to be 640 Hz, when $n = 1$. The length of the string is 33 cm and its mass is 0.125 g. Hence we have

$$\frac{\omega L}{c} = n\pi \quad \Rightarrow \quad \nu_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

$$c = 2L\nu_1 = 2\times 0.33\times 640 = 422.4 \text{ m/s}$$

$$\mu = 0.125\times 10^{-3}/0.33 = 3.8\times 10^{-4} \text{ kg/m}$$

$$T = c^2\mu = 422.4^2\times 3.8\times 10^{-4} = 68 \text{ N}$$

This is a pull of about 7 kg if converted to a mass. Note there are 4 strings.

Wave transmission and boundaries

Consider a string that has a discontinuous change in properties at some point, which we call x = 0.

To the left, we have mass per unit length of μ_1 and velocity of wave c_1, and to the right we have μ_2 and c_2.

We have an incident wave travelling from left to right. Part of this wave is transmitted, and part is reflected. The boundary condition is that the displacement of both parts at x = 0 is identical. You also need to have a continuous derivative or else you form a kink. Hence we have

$$y_1(0) = y_2(0) \quad ; \quad \left.\frac{\partial y_1}{\partial x}\right|_{x=0} = \left.\frac{\partial y_2}{\partial x}\right|_{x=0}$$

Define the three waves, noting the sign of k defines the direction of the wave

$$y_i = A_i\sin(k_1 x - \omega_1 t)$$

$$y_r = A_r\sin(-k_1 x - \omega_1 t)$$

$$y_t = A_t\sin(k_2 x - \omega_1 t)$$

Note that the frequency of the transmitted wave stays the same, but the wavelength changes, because the junction needs to move by both waves at the same frequency.

We have boundary conditions:

$$y_1(0) = y_2(0) \quad \Rightarrow \quad A_i + A_r = A_t$$

$$\left.\frac{\partial y_1}{\partial x}\right|_{x=0} = \left.\frac{\partial y_2}{\partial x}\right|_{x=0} \quad \Rightarrow \quad -A_i k_1 + A_r k_1 = -A_t k_2$$

Too many unknowns as it stands, but we have the ratio of wave vectors from our given facts:

$$\omega = ck \quad \Rightarrow \quad c_1 k_1 = c_2 k_2 \quad \Rightarrow \quad k_1 / k_2 = c_2 / c_1$$

Hence we have

$$(A_i - A_r) c_2 = A_t c_1$$

We can combine these two equations:

$$A_i + A_r = A_t$$

$$A_i - A_r = A_t \frac{c_1}{c_2}$$

$$2A_i = A_t \left(1 + \frac{c_1}{c_2}\right) \quad \Rightarrow \quad A_t = \frac{2c_2}{c_1 + c_2} A_i$$

$$\frac{c_2}{c_1}(A_i - A_r) = A_t$$

$$\frac{c_2}{c_1}(A_i - A_r) - (A_i + A_r) = 0$$

$$c_2 (A_i - A_r) = c_1 (A_i + A_r) \quad \Rightarrow \quad (c_1 + c_2) A_r = (c_2 - c_1) A_i \quad \Rightarrow \quad A_r = \frac{c_2 - c_1}{c_1 + c_2} A_i$$

Example of a fixed end, such that $c_2 = 0$

$$A_t = \frac{2c_2}{c_1 + c_2} A_i = 0$$

$$A_r = \frac{c_2 - c_1}{c_1 + c_2} A_i = \frac{-c_1}{c_1} A_i = -A_i$$

This means the reflected wave goes back with opposite displacement. A peak hits the wall and reflects as a trough. This is the standard reflection condition.

Now take the case that $c_1 = c_2$. It follows that

$$A_t = \frac{2c_2}{c_1 + c_2} A_i = A_i$$

$$A_r = \frac{c_2 - c_1}{c_1 + c_2} A_i = 0$$

Air in a column

Very quickly to note, given that all this has links with musical instruments, that we can think quickly about air in a pipe. We could have open or closed ends.

At an open end, the pressure is zero, so any wave set up in the air will have a maximum at the open end.

At the closed end, nothing moves, so you will have a node.

Thus a pipe will support different harmonics depending on whether the ends are open or closed. Typically one end is open and one closed.

Musical instruments

Here the idea is to exploit forced oscillations, so that you can convert the energy of the primary oscillator (eg a string) into a sound wave that propagates through air. Typically the primary oscillator drives oscillations of another component, such as the body of a violin which in turn forces oscillations of the air to generate sound waves. We have the problem of resonant frequencies, but complex three-dimensional (or two-dimensional) objects will have many resonances, and we recall that resonances can be broad with some damping, so that the resonator will pick up a wide range of frequencies. But note that there will inevitably be an art in designing these resonators and a critical choice of materials.

Note also that musical notes will die away. Better instruments will have a reasonable degree of sustain, ie relative high Q factor. But you can add a light bit of friction, mechanically or by hand, to lower Q to have sharper cut-off of the note.

Summary

1. We have looked at the wave equations in continuous media
2. We have looked at the effect of a boundary in the media