Topic 7: Many coupled oscillators

1. Model

Draw several balls and springs connected to fixed walls at each end.

Note that technically we are talking about motions in the direction of the spring, but the formalism will extend to transverse motions also – draw these – and the formalism is the same.

We won't worry about the differences just yet, but we will draw pictures of displacements in the vertical axis to make things more visual, but we will be considering longitudinal motions.

So we have N balls. How many normal modes will there be? N of course.

So let's guess some solutions.

1. Lowest mode, a single sinusoidal displacement with wavelength equal to twice the distance between walls
2. Next mode, a single sinusoidal displacement with wavelength equal to the distance between walls, that is, twice the distance between walls divided by 2.
3. Third mode, the same with the wavelength equal to twice the 2/3 the distance between walls = twice the distance between walls divided by 3.
4. So now we might have a trend, in that the allowed oscillations look like standing waves with wavelength = twice the distance between walls divided by the order of the wave
5. So how far can we go? The N'th mode has to be a wavelength equal to twice the distance between walls divided by N. Note that the distance between atoms = the distance between walls divided by $N + 1$, so each ball moves sequentially up and down.
6. If there were a next mode up, the wavelength would equal the distance between balls, so each ball would sit at a zero. All other motions will be linear combinations of these waves because shorter wavelengths look like longer wavelengths. We will prove this in the next lecture.

We consider the boundary conditions, namely that the end points, namely ball 0 and ball $N + 1$, cannot move. Hence a general solution for the displacement of any ball j for any mode n at any random time is given as

$$u_{j,n} = A_n \sin\left(\frac{jn\pi}{N+1}\right)$$

Note that we have not included any time dependence – this represents a static displacement of amplitude A_n.

What does this mean?

1. Case $n = 1$, for j between 0 and $N + 1$ the argument ranges between 0 and π, so there is a single maximum half way along the string.

2. Case $n = 2$, the same argument ranges between 0 and 2π, so there is a zero value half way along the string
3. Case n = 3, the argument ranges between 0 and 3π, so we have two zeroes along the string as well as at the end.
4. Case $n = N$, the argument ranges between 0 and $N\pi$, with $N - 1$ zeroes along the string as well as at the end, or $N + 1$ zeroes including the end.. Recall comment about the next highest mode, which would give N zeroes, at the positions of the balls.

Equation of motion

Let us take one ball and its two neighbours. The energy for displacements can be written as

$$E_j = \frac{1}{2}k\left(u_j - u_{j-1}\right)^2 + \frac{1}{2}k\left(u_j - u_{j+1}\right)^2$$

So we can do what we always do and compute the force:

$$F_j = -\frac{\mathrm{d}E_j}{\mathrm{d}u_j} = -k\left(u_j - u_{j-1}\right) - k\left(u_j - u_{j+1}\right) = -k\left(2u_j - u_{j-1} - u_{j+1}\right)$$

So how to proceed? We have already shown that modes that are purely harmonic do not interact, but instead are superimposed. So we can take one mode alone and solve the equation for that mode only.

We take mode n and input the previous equation with now an explicit time dependence:

$$u_{j,n} = A_n \sin\left(\frac{jn\pi}{N+1}\right)\cos(\omega_n t)$$

This gives us

$$F_j = -k\left(2u_j - u_{j-1} - u_{j+1}\right) = -kA_n\left(2\sin\left(\frac{jn\pi}{N+1}\right) - \sin\left(\frac{(j-1)n\pi}{N+1}\right) - \sin\left(\frac{(j+1)n\pi}{N+1}\right)\right)\cos(\omega_n t)$$

Recall that we have the trigonometric identities

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$
$$\sin(a+b) + \sin(a-b) = 2\sin a\cos b$$

Thus we have

$$
\begin{aligned}
F_j &= -kA_n\left(2\sin\left(\frac{jn\pi}{N+1}\right)-\sin\left(\frac{(j-1)n\pi}{N+1}\right)-\sin\left(\frac{(j+1)n\pi}{N+1}\right)\right)\cos(\omega_n t)\\
&= -kA_n\left(2\sin\left(\frac{jn\pi}{N+1}\right)-2\sin\left(\frac{jn\pi}{N+1}\right)\cos\left(\frac{n\pi}{N+1}\right)\right)\cos(\omega_n t)\\
&= -kA_n 2\sin\left(\frac{jn\pi}{N+1}\right)\left(1-\cos\left(\frac{n\pi}{N+1}\right)\right)\cos(\omega_n t)\\
&= -kA_n 2\sin\left(\frac{jn\pi}{N+1}\right)\left(2\sin^2\left(\frac{n\pi}{2(N+1)}\right)\right)\cos(\omega_n t)
\end{aligned}
$$

Now we consider the mass x acceleration term

$$
\begin{aligned}
u_{j,n} &= A_n \sin\left(\frac{jn\pi}{N+1}\right)\cos(\omega_n t)\\
m\ddot{u}_{j,n} &= -m\omega_n^2 \sin\left(\frac{jn\pi}{N+1}\right)\cos(\omega_n t)
\end{aligned}
$$

So let us equate the two

$$
\begin{aligned}
F_j &= -kA_n 2\sin\left(\frac{jn\pi}{N+1}\right)\left(2\sin^2\left(\frac{n\pi}{2(N+1)}\right)\right)\cos(\omega_n t) = -m\omega_n^2 \sin\left(\frac{jn\pi}{N+1}\right)\cos(\omega_n t)\\
\omega_n^2 &= \frac{4k}{m}\sin^2\left(\frac{n\pi}{2(N+1)}\right)\\
\omega_n &= 2\sqrt{\frac{k}{m}}\sin\left(\frac{n\pi}{2(N+1)}\right)\\
&= 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)
\end{aligned}
$$

Example of N = 5. We have

$$
\begin{aligned}
\omega_1 &= 2\omega_0 \sin\left(\frac{\pi}{2\times 6}\right) = 2\omega_0 \sin(15^\circ) = 0.52\omega_0\\
\omega_2 &= 2\omega_0 \sin\left(\frac{2\pi}{2\times 6}\right) = 2\omega_0 \sin(30^\circ) = \omega_0\\
\omega_3 &= 2\omega_0 \sin\left(\frac{3\pi}{2\times 6}\right) = 2\omega_0 \sin(45^\circ) = 1.41\omega_0\\
\omega_4 &= 2\omega_0 \sin\left(\frac{4\pi}{2\times 6}\right) = 2\omega_0 \sin(60^\circ) = 1.73\omega_0\\
\omega_5 &= 2\omega_0 \sin\left(\frac{5\pi}{2\times 6}\right) = 2\omega_0 \sin(75^\circ) = 1.93\omega_0
\end{aligned}
$$

The ratios are not integers, which means that you won't get simple repetition of states if you excite two normal modes.

A string

So let's generalise to the case where the balls are atoms and the total length is say 50 cm. How many atoms? Well, an atom separation is about 10^{-10} m, so we might have the equivalent of 5×10^9 atoms or balls. Note that this is for an infinitesimally thin string, but in reality a string is thick.

So when we have a guitar string, we expect the lowest order modes to have a linear relationship with frequency. This is achieved because the arguments on the sine are small and hence the sine is effectively linear.

Summary

1. We have explored the dynamics of a row of balls fixed at one end, and derived the normal modes and corresponding frequencies
2. This model works in the limit of extremely large number of balls, to the point whereby we can think of the model as representing atoms in a macroscopic string
3. We have been interested in the long wavelength waves; next time we will start to think about what happens when we don't fixe the ends and the wavelength approaches the spacing between atoms.