Topic 6: Coupled oscillators 2

1. Recap from last time

We moved on from looking at single oscillators to look at systems that support several oscillations that are coupled.

These systems have several dynamic variables. For 2 objects in one dimension, we have two variables = the displacements of each object.

We have an equation for each variable, but we noted in our examples that the fundamental pattern of motions was not a single variable. Instead for two objects we found two new variables, which for our examples were the amplitude of in-phase and out-of-phase motions.

We defined the concept of a normal mode:

1. Well-defined pattern of displacements
2. Single frequency
3. Different normal modes do not interact – sometimes called orthogonal, because the vector dot product of motions will be zero. As for the case of in-phase and out-of-phase motions; but beware because when we define different masses we meed to be careful

We recast our simple two coupled oscillators in terms of normal modes and found that the energy could be written in terms of variables that did not couple in the final energy, even though the individual displacement variables are coupled.

We developed a recipe that we will review later after completing the CO_2 example.

2. CO_2 vibrations

Complete example and review of recipe from the end of last topic.

3. Double pendulum

Draw a system of two balls on a single pendulum, both of equal mass and equal separation from the pivot

Define variables x_1,y_1 and x_2,y_2, with 1 being the top ball and 2 as the bottom ball

Energy of the top ball is as before

$$E_1 = mgy_1 = \frac{mg}{2L}x_1^2$$

The second ball will have its energy affected by the displacement of the top ball, because it will have already been lifted by the first ball. We also need to be careful about our definition of x_2. We can write

$$E_2 = mgy_2 = \frac{mg}{2L}x_1^2 + \frac{mg}{2L}(x_2 - x_1)^2$$

So now we have the two forces

$$E = E_1 + E_2 = \frac{mg}{L}x_1^2 + \frac{mg}{2L}(x_2 - x_1)^2$$
$$F_1 = -\frac{\mathrm{d}E}{\mathrm{d}x_1} = -\frac{2mg}{L}x_1 + \frac{mg}{L}(x_2 - x_1) = \frac{mg}{L}(x_2 - 3x_1) = m\ddot{x}_1$$
$$F_2 = -\frac{\mathrm{d}E}{\mathrm{d}x_2} = -\frac{mg}{L}(x_2 - x_1) = m\ddot{x}_2$$
$$\ddot{x}_1 + \omega_0^2(3x_1 - x_2) = 0$$
$$\ddot{x}_2 + \omega_0^2(x_2 - x_1) = 0$$

Now we try general solutions as before

$$x_1 = C_1 \cos\omega t$$
$$x_2 = C_2 \cos\omega t$$

These lead to the equations

$$-C_1\omega^2 + \omega_0^2(3C_1 - C_2) = 0$$
$$-C_2\omega^2 + \omega_0^2(C_2 - C_1) = 0$$

We rewrite these in matrix form as

$$\begin{pmatrix} 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

Recall that this is an eigenvalue problem. We solve the eigenvalue from the following:

$$\begin{vmatrix} 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0$$
$$(3\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - (\omega_0^2)^2 = 0$$
$$= 3(\omega_0^2)^2 - 4\omega_0^2\omega^2 + (\omega^2)^2 - (\omega_0^2)^2 = 0$$
$$= 2(\omega_0^2)^2 - 4\omega_0^2\omega^2 + (\omega^2)^2 = 0$$
$$\omega^2 = \frac{4\omega_0^2 \pm \sqrt{16(\omega_0^2)^2 - 8(\omega_0^2)^2}}{2} = 2\omega_0^2 \pm \sqrt{2}\omega_0^2 = \left(2 \pm \sqrt{2}\right)\omega_0^2$$

Now we can solve also for the relative displacements by substituting in the two equations

First solution

$$\begin{pmatrix} 3\omega_0^2-\left(2-\sqrt{2}\right)\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\left(2-\sqrt{2}\right)\omega_0^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$= \begin{pmatrix} \left(1+\sqrt{2}\right)\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \left(-1+\sqrt{2}\right)\omega_0^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$\left(1+\sqrt{2}\right)C_1 = C_2$$

$$C_1 = \left(-1+\sqrt{2}\right)C_2$$

$$\left(1+\sqrt{2}\right)C_1 = \left(1+\sqrt{2}\right)\left(-1+\sqrt{2}\right)C_2 = \left(-1+\sqrt{2}-\sqrt{2}+2\right)C_2 = C_2$$

$$\frac{C_1}{C_2} = \frac{1}{1+\sqrt{2}}$$

This is the in-phase solution.

Second solution

$$\begin{pmatrix} 3\omega_0^2-\left(2+\sqrt{2}\right)\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\left(2+\sqrt{2}\right)\omega_0^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$= \begin{pmatrix} \left(1-\sqrt{2}\right)\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\left(1+\sqrt{2}\right)\omega_0^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$\left(1-\sqrt{2}\right)C_1 = C_2$$

$$C_1 = -\left(1+\sqrt{2}\right)C_2$$

$$\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)C_1 = \left(1+\sqrt{2}\right)C_2 = \left(1+\sqrt{2}-\sqrt{2}-2\right)C_1 = -C_1$$

$$\frac{C_1}{C_2} = -\left(1+\sqrt{2}\right)$$

And this is the out-of-phase solution.

4. Driven double pendulum

Redraw the double pendulum but now have added displacement of top ball

$$\eta = \eta_0 \cos\omega t$$

Note now that the angular frequency is now imposed and will determine the frequencies of the vibrations. It no longer is an unknown to be solved.

Given that we also know the amplitude of the imposed motion we should be able to solve exactly for our two unknowns, C_1 and C_2.

Note that in what we do now we will forget about damping. To include damping will not add the new insights we are after, but will make the process much more complicated.

The sole change from before is that we now have an additional term for the equation for the top ball, namely we need to consider the rising of the first ball taking account of the position of the pivot of the pendulum.

$$E_1 = \frac{mg}{2L}\left(x_1 - \eta\right)^2$$

$$E = E_1 + E_2 = \frac{mg}{2L}\left(x_1 - \eta\right)^2 + \frac{mg}{2L}\left(x_1 - \eta\right)^2 + \frac{mg}{2L}\left(x_2 - x_1\right)^2$$

$$= \frac{mg}{L}\left(x_1 - \eta\right)^2 + \frac{mg}{2L}\left(x_2 - x_1\right)^2$$

$$F_1 = -\frac{\mathrm{d}E}{\mathrm{d}x_1} = -\frac{2mg}{L}\left(x_1 - \eta\right) + \frac{mg}{L}\left(x_2 - x_1\right) = m\ddot{x}_1$$

$$\ddot{x}_1 + 2\omega_0^2\left(x_1 - \eta\right) - \omega_0^2\left(x_2 - x_1\right) = 0$$

$$\ddot{x}_1 + 3\omega_0^2 x_1 - \omega_0^2 x_2 = 2\omega_0^2 \eta_0 \cos\omega t$$

From before we also had

$$\ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0$$

Now we use the same solutions as before and obtain

$$-C_1\omega^2 + \omega_0^2(3C_1 - C_2) = 0$$

$$-C_2\omega^2 + \omega_0^2(C_2 - C_1) = 0$$

$$-C_1\omega^2 + 3\omega_0^2 C_1 - \omega_0^2 C_2 = 2\omega_0^2\eta_0$$

$$-C_2\omega^2 + \omega_0^2 C_2 - \omega_0^2 C_1 = 0$$

Put this in matrix form to obtain

$$\begin{pmatrix} 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 2\omega_0^2\eta_0 \\ 0 \end{pmatrix}$$

Which we can tackle by inverting the matrix

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 3\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{pmatrix}^{-1} \times \begin{pmatrix} 2\omega_0^2\eta_0 \\ 0 \end{pmatrix}$$

Recall the inverse of a 2 x 2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This gives us

$$\begin{pmatrix} 3\omega_0^2-\omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\omega^2 \end{pmatrix}^{-1} = \frac{1}{\left(3\omega_0^2-\omega^2\right)\left(\omega_0^2-\omega^2\right)-\left(\omega_0^2\right)^2}\begin{pmatrix} \omega_0^2-\omega^2 & \omega_0^2 \\ \omega_0^2 & 3\omega_0^2-\omega^2 \end{pmatrix}$$

$$\left(3\omega_0^2-\omega^2\right)\left(\omega_0^2-\omega^2\right)-\left(\omega_0^2\right)^2 = 2\left(\omega_0^2\right)^2-4\omega^2\omega_0^2+\left(\omega^2\right)^2$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{2\left(\omega_0^2\right)^2-4\omega^2\omega_0^2+\left(\omega^2\right)^2}\begin{pmatrix} \left(\omega_0^2-\omega^2\right)2\omega_0^2\eta_0 \\ 2\left(\omega_0^2\right)^2\eta_0 \end{pmatrix}$$

Let's look at the denominator

$$2\left(\omega_0^2\right)^2-4\omega^2\omega_0^2+\left(\omega^2\right)^2 = \left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_2^2\right)$$

$$\omega_1^2+\omega_2^2 = 4\omega_0^2$$

$$\omega_1^2\omega_2^2 = 2\left(\omega_0^2\right)^2$$

$$\omega_1^2 = 2\omega_0^2-\delta \quad ; \quad \omega_2^2 = 2\omega_0^2+\delta$$

$$\omega_1^2\omega_2^2 = 4\omega_0^2-\delta^2 \quad \Rightarrow \quad \delta^2 = 2\left(\omega_0^2\right)^2 \quad ; \quad \delta = \sqrt{2}\omega_0^2$$

$$\omega_1^2 = \left(2-\sqrt{2}\right)\omega_0^2 \quad ; \quad \omega_2^2 = \left(2+\sqrt{2}\right)\omega_0^2$$

Note that these are the same as the two normal mode frequencies. No surprise but nice!

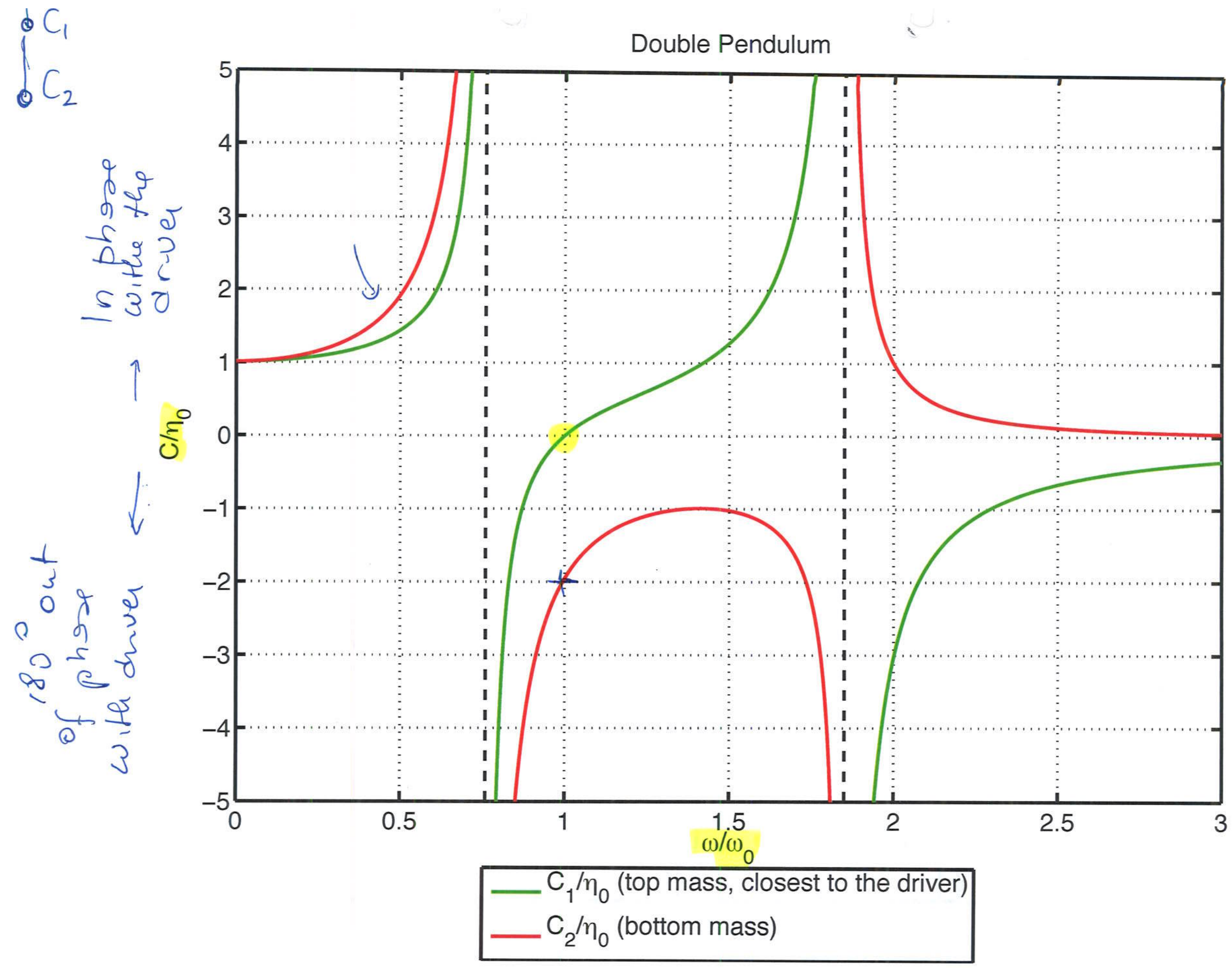
Thus we have

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{\left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_2^2\right)}\begin{pmatrix} \left(\omega_0^2-\omega^2\right)2\omega_0^2\eta_0 \\ 2\left(\omega_0^2\right)^2\eta_0 \end{pmatrix}$$

$$C_1 = \frac{\left(\omega_0^2-\omega^2\right)2\omega_0^2\eta_0}{\left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_2^2\right)}$$

$$C_2 = \frac{2\left(\omega_0^2\right)^2\eta_0}{\left(\omega^2-\omega_1^2\right)\left(\omega^2-\omega_2^2\right)}$$

So draw the graph in stages …



Note that the resonant frequencies are

$$\omega_1 = \sqrt{2-\sqrt{2}}\,\omega_0 \simeq 0.77\omega_0$$
$$\omega_2 = \sqrt{2+\sqrt{2}}\,\omega_0 \simeq 1.85\omega_0$$

Draw axes and mark in the resonant frequencies. Note the following …

1. At near-zero applied frequency we expect both balls to move together with the imposed motion

$$C_1 = \frac{2\omega_0^4\eta_0}{\omega_1^2\omega_2^2} = \frac{2\omega_0^4\eta_0}{2\omega_0^4} = \eta_0 = C_2$$

2. On increase applied frequency, denominator becomes smaller, and for C_1 numerator also becomes smaller but not as fast. Draw the first part. Note that both C_1 and C_2 are the same sign, and the same sign as the applied displacement.

3. As the applied frequency tends towards the first resonant frequency, the denominator tends to zero so we have a divergence of both displacements (still in phase)

4. At just above the first resonant frequency ω_1, the denominator is now negative in both cases, but again coming down from minus infinity.

5. When we hit ω_0 the numerator for C_1 goes to zero, and beyond that the numerator is negative so that C_1 is positive. C_2 always remains negative.

6. At this point where $C_1 = 0$, for C_2 we have

$$C_2 = \frac{2\left(\omega_0^2\right)^2 \eta_0}{\left(\omega_0^2 - \omega_1^2\right)\left(\omega_0^2 - \omega_2^2\right)} = \frac{2\left(\omega_0^2\right)^2 \eta_0}{\left(\omega_0^2\right)^2\left(1 - 2 + \sqrt{2}\right)\left(1 - 2 - \sqrt{2}\right)} = \frac{2\eta_0}{-\left(\sqrt{2} - 1\right)\left(1 + \sqrt{2}\right)} = -2\eta_0$$

Draw this mode. Some degree of incredulity. But note that we do have to have a cross-over from in-phase to out-of-phase motion, and this is the point where this happens.

7. So as we moved towards the second resonance, C_1 heads to +infinity and C_2 heads to minus infinity. As we head towards this second resonance, we have the out-of-phase motions.

8. On increasing the frequency beyond the first resonance, the denominators become larger and larger with positive value. But note that the numerator for C_1 is negative and the numerator for C_2 is always positive, so now we get a full switch coming away from the resonance, and on increasing applied frequency the two amplitudes head towards zero.

5. Summary

The results for this case are quite general

1. For any system we will have resonant frequencies for all the normal modes
2. If the applied force can move things in some way in relationship to the normal mode, you will get resonant behaviour
3. In the case of CO_2, you will get absorption of electromagnetic radiation for the mode in which the C and O move in opposite direction, but not for the symmetric stretch. In more complex cases you might hit a number of resonances, but it does depend on the symmetry of the vibration. In terms of absorption of electromagnetic radiation – usually infrared – the normal mode needs to involve a change in dipole moment
4. For other systems, you can get a range of normal modes excited. For example, the earth will vibrate at a wide range of frequencies when there is an earthquake