Topic 5: Coupled oscillators

1. Physical example

Show example of two pendulums on a string. Move both and note chaotic. Move one and watch it slow down and the other start up.

Post question: Can we unravel this?

2. Worked example

Two pendulums with spring connection

Draw two types of motion, in phase and exactly out of phase.

Note that we are not going to consider damping at all – will make things more complicated and mask the things we want to learn.

In phase motion will not stretch the spring, and is therefore easily predicted from previous theory of the pendulum

Recall that we have

$$E = \frac{mg}{2L}x^2$$

$$F = -\frac{\mathrm{d}E}{\mathrm{d}x} = -\frac{mg}{L}x = m\ddot{x}$$

$$m\ddot{x} + \frac{mg}{L}x = 0$$

$$x = A\cos\omega t$$

$$\omega = \sqrt{\frac{g}{L}}$$

This will be the frequency for this mode.

$$\omega_1 = \sqrt{\frac{g}{L}}$$

Second mode stretches the spring, so will have a higher frequency. Now write the energy as

$$E = \frac{mg}{2L}x_a^2 + \frac{mg}{2L}x_b^2 + \frac{1}{2}k\left(x_a - x_b\right)^2$$

$$F_a = -\frac{\mathrm{d}E}{\mathrm{d}x_a} = -\frac{mg}{L}x_a - k\left(x_a - x_b\right) = m\ddot{x}_a$$

$$x_a = -x_b$$

$$m\ddot{x}_a + \frac{mg}{L}x_a + 2kx_a = 0$$

$$\ddot{x} + \left(\frac{g}{L} + 2\omega_S^2\right)x = 0$$

$$x = A\cos\omega t$$

$$\omega = \sqrt{\frac{g}{L} + 2\omega_S^2}$$

Question, what happens if I move the spring up? Take the limit that it moves to the top, where you end up with a single pendulum. Hence moving the spring up will reduce the frequency.

3. Normal modes

These two vibrations can be called "normal modes". Features of normal modes are

1. All objects move with a single frequency

2. All objects move with a definite phase relation, in phase or out of phase

3. Normal modes do not interact, but we aren't there yet

So let's define two normal mode coordinates, and rewrite the energy

$$Q_1 = \frac{1}{\sqrt{2}}\left(x_a + x_b\right)$$

$$Q_2 = \frac{1}{\sqrt{2}}\left(x_a - x_b\right)$$

$$\frac{1}{\sqrt{2}}\left(Q_1 + Q_2\right) = x_a$$

$$\frac{1}{\sqrt{2}}\left(Q_1 - Q_2\right) = x_b$$

$$\begin{aligned}
E &= \frac{mg}{2L}x_a^2 + \frac{mg}{2L}x_b^2 + \frac{1}{2}k\left(x_a - x_b\right)^2 \\
&= \frac{mg}{4L}\left(Q_1 + Q_2\right)^2 + \frac{mg}{4L}\left(Q_1 - Q_2\right)^2 + kQ_2^2 \\
&= \frac{mg}{4L}\left(Q_1^2 + 2Q_1Q_2 + Q_2^2\right) + \left(Q_1^2 - 2Q_1Q_2 + Q_2^2\right) + kQ_2^2 \\
&= \frac{mg}{2L}\left(Q_1^2 + Q_2^2\right) + \frac{1}{2}kQ_2^2
\end{aligned}$$

So we see that there is no term in the energy that involves both normal mode coordinates. Good!

Note that we have two normal modes because we have two degrees of freedom. This result can be generalised.

4. Understanding effects of initial conditions

I need to decide how I am going to set this up, that is what are my initial conditions. Let me move one of these and start both with zero velocity. Thus I might write

$$
\begin{aligned}
x_a &= C_{1a}\cos\omega_1 t + C_{2a}\cos\omega_2 t \\
x_b &= C_{1b}\cos\omega_1 t + C_{2b}\cos\omega_2 t \\
\dot{x}_a &= -\omega_1 C_{1a}\sin\omega_1 t - \omega_2 C_{2a}\sin\omega_2 t \\
\dot{x}_b &= -\omega_1 C_{1b}\sin\omega_1 t - \omega_2 C_{2b}\sin\omega_2 t
\end{aligned}
$$

$$@t = 0$$

$$
\begin{aligned}
x_a &= C_{1a} + C_{2a} \\
x_b &= 0 = C_{1b} + C_{2b} \Rightarrow C_{1b} = -C_{2b} \\
\dot{x}_a &= 0 \\
\dot{x}_b &= 0
\end{aligned}
$$

Note that this confirms we need not add phases to the cosines. We know from before that for one mode the C values are the same, and for the other they are opposite. So we can write

$$
\begin{aligned}
x_a &= C\cos\omega_1 t + C\cos\omega_2 t \\
x_b &= C\cos\omega_1 t - C\cos\omega_2 t
\end{aligned}
$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$
\begin{aligned}
x_a &= 2C\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) \\
x_b &= 2C\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)
\end{aligned}
$$

Note we get beating, with and slow parts. At time = 0 only x_a has amplitude. At some point the cosine goes to zero and the sine is a maximum, which means that as the first component stops the second one starts.

So now we understand this complex motion!

5. Doing this formally

$$E = \frac{mg}{2L}x_a^2 + \frac{mg}{2L}x_b^2 + \frac{1}{2}k\left(x_a - x_b\right)^2$$
$$F_a = -\frac{\mathrm{d}E}{\mathrm{d}x_a} = -\frac{mg}{L}x_a - k\left(x_a - x_b\right) = m\ddot{x}_a$$
$$F_b = -\frac{\mathrm{d}E}{\mathrm{d}x_b} = -\frac{mg}{L}x_b + k\left(x_a - x_b\right) = m\ddot{x}_b$$

$$\ddot{x}_a + \omega_0^2 x_a + \omega_S^2\left(x_a - x_b\right) = 0$$
$$\ddot{x}_b + \omega_0^2 x_b - \omega_S^2\left(x_a - x_b\right) = 0$$

Try a general solution:

$$x_a = C_a \cos\omega t$$
$$x_a = C_b \cos\omega t$$

So we put these into our equation, noting we can drop the cosine term because it appears everywhere

$$-\omega^2 C_a + \omega_0^2 C_a + \omega_S^2\left(C_a - C_b\right) = -\omega^2 C_a + \left(\omega_0^2 + \omega_S^2\right)C_a - \omega_S^2 C_b = 0$$
$$-\omega^2 C_b + \omega_0^2 C_b - \omega_S^2\left(C_a - C_b\right) = -\omega^2 C_b + \left(\omega_0^2 + \omega_S^2\right)C_b - \omega_S^2 C_a = 0$$

So where do we go with this? We have two equations but three unknowns. Not good.

But note that we can scale all the C's with a common factor so maybe we should only worry about relative sizes.

Move everything to one side

$$\left(-\omega^2 + \omega_0^2 + \omega_S^2\right)C_a = \omega_S^2 C_b$$
$$\left(-\omega^2 + \omega_0^2 + \omega_S^2\right)C_b = \omega_S^2 C_a$$

$$\frac{C_a}{C_b} = \frac{\omega_S^2}{-\omega^2 + \omega_0^2 + \omega_S^2} = \frac{-\omega^2 + \omega_0^2 + \omega_S^2}{\omega_S^2}$$
$$\left(\omega_S^2\right)^2 = \left(-\omega^2 + \omega_0^2 + \omega_S^2\right)^2$$
$$\pm\omega_S^2 = -\omega^2 + \omega_0^2 + \omega_S^2$$
$$\omega^2 = \omega_0^2 + \omega_S^2 \mp \omega_S^2$$
$$\omega_1^2 = \omega_0^2$$
$$\omega_2^2 = \omega_0^2 + 2\omega_S^2$$

Exactly as before. No surprises, but nice.

For the two solutions

$$\frac{C_{1a}}{C_{1b}} = \frac{-\omega_0^2 + \omega_0^2 + \omega_S^2}{\omega_S^2} = \frac{\omega_S^2}{\omega_S^2} = +1$$

$$\frac{C_{2a}}{C_{2b}} = \frac{-\left(\omega_0^2 + 2\omega_S^2\right) + \omega_0^2 + \omega_S^2}{\omega_S^2} = \frac{-\omega_S^2}{\omega_S^2} = -1$$

Which is what we were expecting.

But we could have done this a slightly different way, which is going to be more useful going forward.

$$-\omega^2 C_a + \left(\omega_0^2 + \omega_S^2\right) C_a - \omega_S^2 C_b = 0$$

$$-\omega^2 C_b + \left(\omega_0^2 + \omega_S^2\right) C_b - \omega_S^2 C_a = 0$$

$$\begin{pmatrix} \left(\omega_0^2 + \omega_S^2\right) - \omega^2 & -\omega_S^2 \\ -\omega_S^2 & \left(\omega_0^2 + \omega_S^2\right) - \omega^2 \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = 0$$

This is an eigenvalue/eigenvector problem. The solution exists for

$$\begin{vmatrix} \left(\omega_0^2 + \omega_S^2\right) - \omega^2 & -\omega_S^2 \\ -\omega_S^2 & \left(\omega_0^2 + \omega_S^2\right) - \omega^2 \end{vmatrix} = 0$$

$$\left(\left(\omega_0^2 + \omega_S^2\right) - \omega^2\right)^2 = \left(\omega_S^2\right)^2$$

Exactly as before!

But now we can work out the eigenvectors.

$$\begin{pmatrix} \left(\omega_0^2 + \omega_S^2\right) - \omega_0^2 & -\omega_S^2 \\ -\omega_S^2 & \left(\omega_0^2 + \omega_S^2\right) - \omega_0^2 \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = \begin{pmatrix} \omega_S^2 & -\omega_S^2 \\ -\omega_S^2 & \omega_S^2 \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = 0 \Rightarrow C_{1a} = C_{1b}$$

$$\begin{pmatrix} \left(\omega_0^2 + \omega_S^2\right) - \left(\omega_0^2 + 2\omega_S^2\right) & -\omega_S^2 \\ -\omega_S^2 & \left(\omega_0^2 + \omega_S^2\right) - \left(\omega_0^2 + 2\omega_S^2\right) \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = \begin{pmatrix} -\omega_S^2 & -\omega_S^2 \\ -\omega_S^2 & -\omega_S^2 \end{pmatrix} \times \begin{pmatrix} C_a \\ C_b \end{pmatrix} = 0 \Rightarrow C_{1a} = -C_{1b}$$

This is the approach I recommend

6. Example of CO_2 molecule

Now we have three atoms, which can move along the axis (ignore transverse motions for now. Energy is given as

$$E=\frac{1}{2}k\left(x_{\mathrm{O1}}-x_{\mathrm{C}}\right)^2+\frac{1}{2}k\left(x_{\mathrm{O2}}-x_{\mathrm{C}}\right)^2$$
$$F_{\mathrm{O1}}=-\frac{\mathrm{d}E}{\mathrm{d}x_{\mathrm{O1}}}=-k\left(x_{\mathrm{O1}}-x_{\mathrm{C}}\right)=m_{\mathrm{O}}\ddot{x}_{\mathrm{O1}}$$
$$F_{\mathrm{O2}}=-\frac{\mathrm{d}E}{\mathrm{d}x_{\mathrm{O2}}}=-k\left(x_{\mathrm{O2}}-x_{\mathrm{C}}\right)=m_{\mathrm{O}}\ddot{x}_{\mathrm{O2}}$$
$$F_{\mathrm{C}}=-\frac{\mathrm{d}E}{\mathrm{d}x_{\mathrm{C}}}=-k\left(2x_{\mathrm{C}}-x_{\mathrm{O1}}-x_{\mathrm{O2}}\right)=m_{\mathrm{C}}\ddot{x}_{\mathrm{C}}$$

Let’s write some general solutions

$$x_{\mathrm{O1}}=C_{\mathrm{O1}}\cos\omega t$$
$$x_{\mathrm{O2}}=C_{\mathrm{O2}}\cos\omega t$$
$$x_{\mathrm{C}}=C_{\mathrm{C}}\cos\omega t$$

So we substitute into the force equations, noting that we can drop the cosine terms again

$$-m_{\mathrm{O}}C_{\mathrm{O1}}\omega^2+k\left(C_{\mathrm{O1}}-C_{\mathrm{C}}\right)=0$$
$$-m_{\mathrm{O}}C_{\mathrm{O2}}\omega^2+k\left(C_{\mathrm{O2}}-C_{\mathrm{C}}\right)=0$$
$$-m_{\mathrm{C}}C_{\mathrm{C}}\omega^2+k\left(2C_{\mathrm{C}}-C_{\mathrm{O1}}-C_{\mathrm{O2}}\right)=0$$
$$\begin{pmatrix} k/m_{\mathrm{O}}-\omega^2 & 0 & -k/m_{\mathrm{O}} \\ 0 & k/m_{\mathrm{O}}-\omega^2 & -k/m_{\mathrm{O}} \\ -k/m_{\mathrm{C}} & -k/m_{\mathrm{C}} & 2k/m_{\mathrm{C}}-\omega^2 \end{pmatrix}\times\begin{pmatrix} C_{\mathrm{O1}} \\ C_{\mathrm{O2}} \\ C_{\mathrm{C}} \end{pmatrix}=0$$

Solutions are obtained from

$$\begin{vmatrix} k/m_{\mathrm{O}}-\omega^2 & 0 & -k/m_{\mathrm{O}} \\ 0 & k/m_{\mathrm{O}}-\omega^2 & -k/m_{\mathrm{O}} \\ -k/m_{\mathrm{C}} & -k/m_{\mathrm{C}} & 2k/m_{\mathrm{C}}-\omega^2 \end{vmatrix}=0$$

From which we obtain

$$\begin{vmatrix} k/m_{\mathrm{O}}-\omega^2 & 0 & -k/m_{\mathrm{O}} \\ 0 & k/m_{\mathrm{O}}-\omega^2 & -k/m_{\mathrm{O}} \\ -k/m_{\mathrm{C}} & -k/m_{\mathrm{C}} & 2k/m_{\mathrm{C}}-\omega^2 \end{vmatrix}=0$$
$$\Rightarrow\left(k/m_{\mathrm{O}}-\omega^2\right)^2\left(2k/m_{\mathrm{C}}-\omega^2\right)-2\left(k/m_{\mathrm{O}}-\omega^2\right)k^2/m_{\mathrm{O}}m_{\mathrm{C}}=0$$

One solution is

$$k/m_{\mathrm{O}}-\omega^2=0$$
$$\omega^2=k/m_{\mathrm{O}}$$

We can divide out this solution to yield

$$\left(k/m_\mathrm{O}-\omega^2\right)\left(2k/m_\mathrm{C}-\omega^2\right)-2k^2/m_\mathrm{O}m_\mathrm{C}=0$$

$$\omega^4-k\left(1/m_\mathrm{O}+2/m_\mathrm{C}\right)\omega^2+2k^2/m_\mathrm{O}m_\mathrm{C}-2k^2/m_\mathrm{O}m_\mathrm{C}=0$$

$$\omega^4-k\left(1/m_\mathrm{O}+2/m_\mathrm{C}\right)\omega^2=0$$

$$\omega^2=0$$

$$\omega^2=k\left(1/m_\mathrm{O}+2/m_\mathrm{C}\right)$$

What about eigenvectors. Take the zero solution

$$\begin{pmatrix} k/m_\mathrm{O} & 0 & -k/m_\mathrm{O} \\ 0 & k/m_\mathrm{O}^2 & -k/m_\mathrm{O} \\ -k/m_\mathrm{C} & -k/m_\mathrm{C} & 2k/m_\mathrm{C} \end{pmatrix} \times \begin{pmatrix} C_\mathrm{O1} \\ C_\mathrm{O2} \\ C_\mathrm{C} \end{pmatrix} = 0$$

$$k/m_\mathrm{O}\times\left(C_\mathrm{O1}-C_\mathrm{C}\right)=0$$

$$k/m_\mathrm{O}\times\left(C_\mathrm{O2}-C_\mathrm{C}\right)=0$$

$$-k/m_\mathrm{C}\times\left(C_\mathrm{O1}+C_\mathrm{O2}\right)+2k/m_\mathrm{C}\times C_\mathrm{C}$$

$$C_\mathrm{C}=C_\mathrm{O1}=C_\mathrm{O2}$$

This is the uniform translation of the molecule

Take the first solution we found

$$\begin{pmatrix} k/m_\mathrm{O}-k/m_\mathrm{O} & 0 & -k/m_\mathrm{O} \\ 0 & k/m_\mathrm{O}-k/m_\mathrm{O} & -k/m_\mathrm{O} \\ -k/m_\mathrm{C} & -k/m_\mathrm{C} & 2k/m_\mathrm{C}-k/m_\mathrm{O} \end{pmatrix} \times \begin{pmatrix} C_\mathrm{O1} \\ C_\mathrm{O2} \\ C_\mathrm{C} \end{pmatrix} = 0$$

$$-k/m_\mathrm{O}\times C_\mathrm{C}=0\Rightarrow C_\mathrm{C}=0$$

$$-k/m_\mathrm{C}\times\left(C_\mathrm{O1}+C_\mathrm{O2}\right)=0\Rightarrow C_\mathrm{O1}=-C_\mathrm{O2}$$

This is the symmetric stretch – draw picture

The other solution is more complicated

$$\begin{pmatrix} -2k/m_\mathrm{C} & 0 & -k/m_\mathrm{O} \\ 0 & -2k/m_\mathrm{C} & -k/m_\mathrm{O} \\ -k/m_\mathrm{C} & -k/m_\mathrm{C} & -k/m_\mathrm{O} \end{pmatrix} \times \begin{pmatrix} C_\mathrm{O1} \\ C_\mathrm{O2} \\ C_\mathrm{C} \end{pmatrix} = 0$$

$$2C_\mathrm{O1}/m_\mathrm{C}=-C_\mathrm{C}/m_\mathrm{O}$$

$$2C_\mathrm{O2}/m_\mathrm{C}=-C_\mathrm{C}/m_\mathrm{O}$$

$$\Rightarrow C_\mathrm{O1}=C_\mathrm{O2}$$

For this mode, the oxygen atoms move one way and the carbon atom moves the other way, preserving the centre of mass – draw picture

Note that we haven't included the transverse motions, which would lead to two additional centre of mass motions and 2 rotations, and 2 more distortions (orthogonal), making a total of 9 modes.

7. Summary of general recipe

1. Write down the energy equation, and differentiate to get forces on all particles. This works better for me that trying to write down forces immediately.
2. Make sure you get the signs right or else it will go horribly wrong!
3. Equate the force to a mass times acceleration
4. Assume for one mode a general solution with a single frequency but a different phase for each particle – the sign of the phase will come in the sign of the relative amplitude
5. Write the equation as a matrix equation, and the set of frequencies are found as the eigenvalues
6. You then obtain the relative motions as the eigenvectors – note that it is relative motion you get; the absolute might be set by the starting conditions
7. For more complex systems, you need to solve the equations on a computer!