Topic 4: Forced oscillators 2

1. Recap

From King chapter 3, which we will continue today

What have we achieved so far…

1. Basic equations

Draw the spring system and remind people of the basic parameters

Differential equation for damped oscillator with addition of force (note that the lack of damping gives an unreasonable resonance):

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \omega_0^2 a \exp(i\omega t)$$
$$a = F / k$$

The way we solved this was by assuming a general solution of the form

$$x = A(\omega)\exp\big(i(\omega t - \delta)\big)$$

We needed to define a phase because we had no prior knowledge that the movement of the object would follow exactly in phase with that of the force.

This gave us a master equation, removing the bits that propagate everywhere, of the form

$$-\omega^2 A(\omega)\exp\big(-i\delta\big) + i\gamma\omega A(\omega)\exp\big(-i\delta\big) + \omega_0^2 A(\omega)\exp\big(-i\delta\big) = \omega_0^2 a$$
$$A(\omega)\big(-\omega^2 + i\gamma\omega + \omega_0^2\big) = \omega_0^2 a \exp(+i\delta)$$

We unpacked the real and imaginary parts. Squaring and adding gave us a general solution for both the amplitude:

$$A(\omega) = \frac{\omega_0^2 a}{\Big(\big(\omega_0^2 - \omega^2\big)^2 + \gamma^2\omega^2\Big)^{1/2}}$$

Then dividing one by the other gave an equation for the phase angle:

$$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

2. Exploring special cases

We explored three regions

a) at low frequency we have

$$A(\omega \to 0) = \frac{\omega_0^2 a}{\left(\left(\omega_0^2 - \cancel{\omega^2}\right)^2 + \gamma^2 \cancel{\omega^2}\right)^{1/2}} = a$$

$$\tan\delta = \frac{\gamma \cancel{\omega}}{\omega_0^2 - \omega^2} = 0$$

So the object moves in phase with the force, and its amplitude is *a*. Remember that the harmonic force is F = kx, so our displacement at any point is given by the force applied.

The zero phase angle reflects the fact that at low frequency we just move the object in line with the force.

b) at the resonant frequency

Here we get the result

$$A(\omega \to \omega_0) = \frac{\omega_0^2 a}{\left(\left(\omega_0^2 - \omega_0^2\right)^2 + \gamma^2 \omega_0^2\right)^{1/2}} = \frac{\omega_0 a}{\gamma} = aQ$$

$$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2} = \frac{\gamma\omega_0}{0} = \infty \Rightarrow \delta = \pi / 2$$

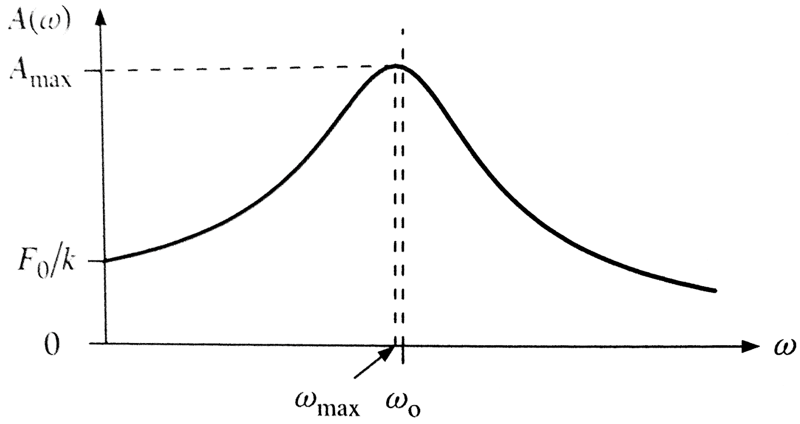
c) Very high frequency

$$A(\omega \to \infty) = \frac{\omega_0^2 a}{\left(\infty + \gamma^2 \omega_0^2\right)^{1/2}} = 0$$

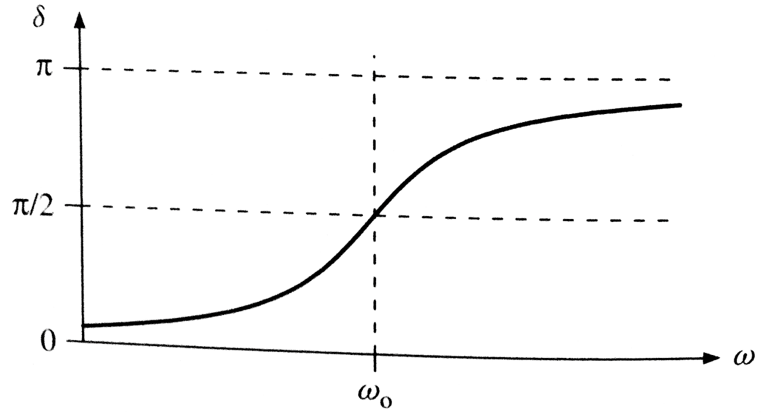
$$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2} = \frac{\gamma\infty}{-\infty^2} = -\frac{\gamma}{\infty} \Rightarrow \delta = \pi$$

So the amplitude dies away at high frequency, but the motions become exactly out of phase

Recall the general picture:



(a)



(b)

It can be shown that

$$\omega_{max}^2 = \omega_0^2 - \gamma^2 / 2$$

which for many systems is not so far away from the resonant frequency.

We also thought of a number of examples of resonance. What we now know is that we can control the behaviour in steady state by ensuring low Q to reduce the amplitude, and also we can choose to build our oscillating artefact from materials that give a resonant frequency away from the likely drive frequency.

Example of millenium bridge – people caused effect by moving with the resonant frequency. Added dampers, show picture

Example of a child's swing, where you add energy at exactly the nature frequency.

Example also of atom, where you can excite effective oscillations into excited states.

Example of light or neutrons exciting lattice vibrations

2. Transient behaviour

Our solutions so far give us no freedom at all, that is given the details of the applied force, what happens next appears to have no relation to the initial conditions. This doesn't really make sense, but it follows directly from the equations we have developed.

So what have we left out? This can be seen by comparing the two equations we have:

$$\ddot{x}+\gamma\dot{x}+\omega_0^2x=\omega_0^2a\exp(i\omega t)$$
$$\ddot{x}+\gamma\dot{x}+\omega_0^2x=0$$

These two have different solutions:

$$\ddot{x}+\gamma\dot{x}+\omega_0^2x=\omega_0^2a\exp(i\omega t)\Rightarrow x=A(\omega)\exp\big(i(\omega t-\delta)\big)$$
$$\ddot{x}+\gamma\dot{x}+\omega_0^2x=0\Rightarrow x=A_0\exp(-\gamma t/2)\exp\big(i(\omega' t+\alpha)\big)\quad;\quad \omega'^2=\omega_0^2-\gamma^2/4$$

Given that the second equation is essentially a zero, we can add zero to both sides to give a general solution of the form

$$x=A(\omega)\exp\big(i(\omega t-\delta)\big)+A_0\exp(-\gamma t/2)\exp\big(i(\omega' t+\alpha)\big)$$

This solution has to work for our forced oscillator differential equation because the latter term leads to zero when operated on by the differential equation.

Note that the second solution is operating at an angular frequency we have no control over – one that the system wants – whereas the first solution is operating at the angular frequency we choose.

The second solution also decays with time, leaving the first solution, which we call the steady state equation. The steady state kicks in after a time determined by the value of γ. The second solution is called the transient, because it lasts for only a finite length of time.

So now we have some starting points. We can define initial conditions in terms of the initial displacement and velocity, such as zero for both, and that will give us values for A_0 and α. But note that will be a bit of a grind.

$$0=A(\omega)\exp(-i\delta)+A_0\exp(i\alpha)$$
$$\begin{aligned}0&=i\omega A(\omega)\exp(-i\delta)-\frac{\gamma}{2}A_0\exp(i\alpha)+i\omega' A_0\exp(i\alpha)\\&=\omega A(\omega)\exp(-i\delta)+A_0\exp(i\alpha)(i\omega'-\gamma/2)\end{aligned}$$

The important point is that now we have allowed the initial conditions to play some sort of role.

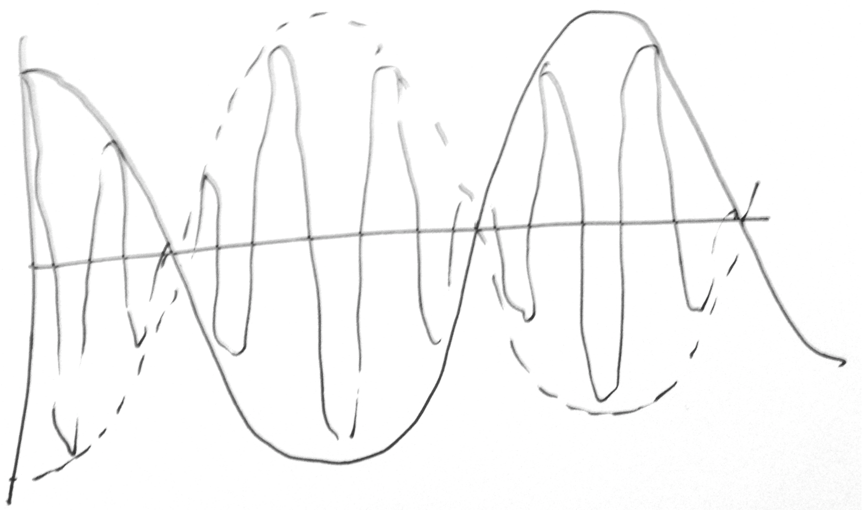
The question then is what happens at the outset? We need to look at these terms. We find that we have the sum of two cosine terms.

So let us consider two cosine terms of very similar angular frequency. We have

$$\cos\omega t+\cos\omega' t=2\cos\left(\frac{\omega+\omega'}{2}t\right)\cos\left(\frac{\omega-\omega'}{2}t\right)$$

For similar frequencies, we have one term multiplied by a much faster term

So you can see that there is a fast signal whose amplitude goes up and down. If this was an audio signal, you would hear this as a tone that beats on and off



Example of tuning guitar

Note that for a guitar, we tune the B string as 247 Hz and E as 330 Hz. Ratio of 4 to 3.

Guitar strings excite higher-order vibrations, in integer steps.

So if I put my finger on the string half way along the E string, I excite the second harmonic at twice the frequency. If I go for 1/3 way along, I excite 3 x 330 Hz = 990 Hz. If I go for 1/4 way along the B string, I excite 4 x 247 = 988 Hz (note rounding). I can tune these strings by listening to the beats.

Power response of a forced oscillator

Energy change is force times displacement (dot product for vectors), and power is the time differential of the energy. In this case, we calculate the time derivative of the position and obtain power as force times velocity

So we develop the equations:

$$x = A(\omega)\cos(\omega t - \delta)$$

$$A(\omega) = \frac{a\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\dot{x} = -A(\omega)\omega\sin(\omega t - \delta)$$

$$P = F\dot{x} \propto \cos\omega t \times \omega A(\omega)\sin(\omega t - \delta)$$

$$\sin(\omega t - \delta) = \sin\omega t\cos\alpha - \cos\omega t\sin\alpha$$

$$\cos\omega t \times \sin(\omega t - \delta) = \cos\omega t\sin\omega t\cos\alpha - \cos^2\omega t\sin\alpha$$

We need the average, not the absolute. The first term averages to zero, and the second to 1/2 times the sine term.

From before we have the sine term from the imaginary part:

$$A(\omega)\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) = \omega_0^2 a\exp(+i\delta)$$

$$A(\omega)\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) = \omega_0^2 a\left(\cos\delta + i\sin\delta\right)$$

$$A(\omega)\left(\omega_0^2 - \omega^2\right) = \omega_0^2 a\cos\delta$$

$$A(\omega)\gamma\omega = \omega_0^2 a\sin\delta$$

$$\Rightarrow \sin\delta = \frac{A(\omega)\gamma\omega}{\omega_0^2 a}$$

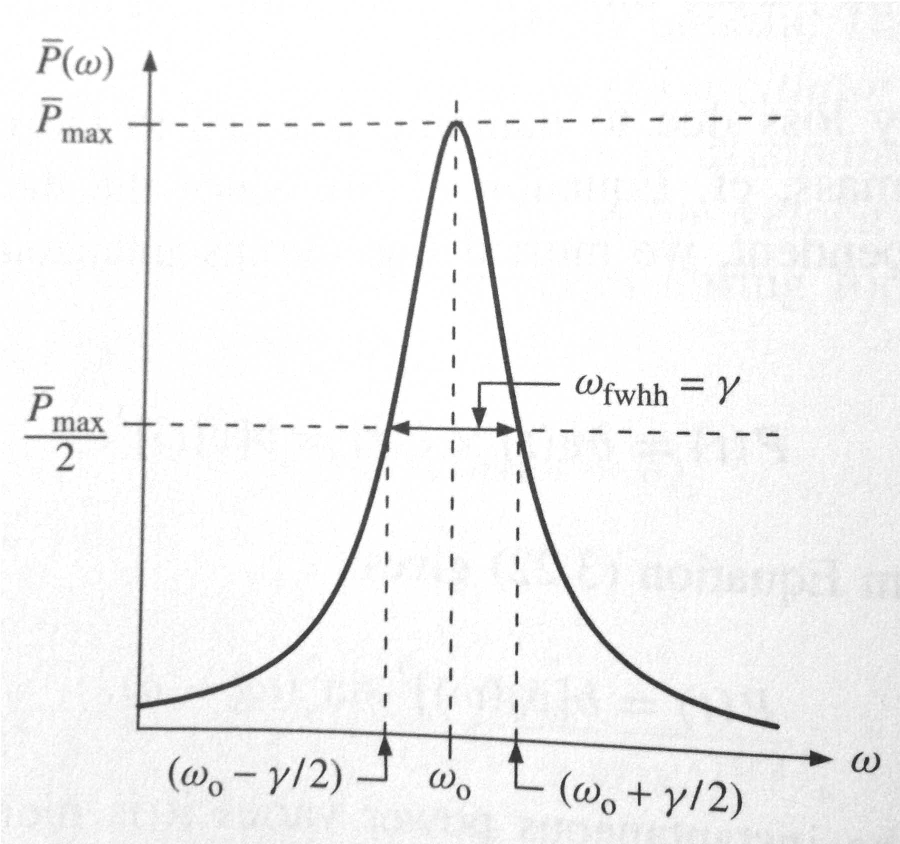
So if we substitute for the sine term, we have

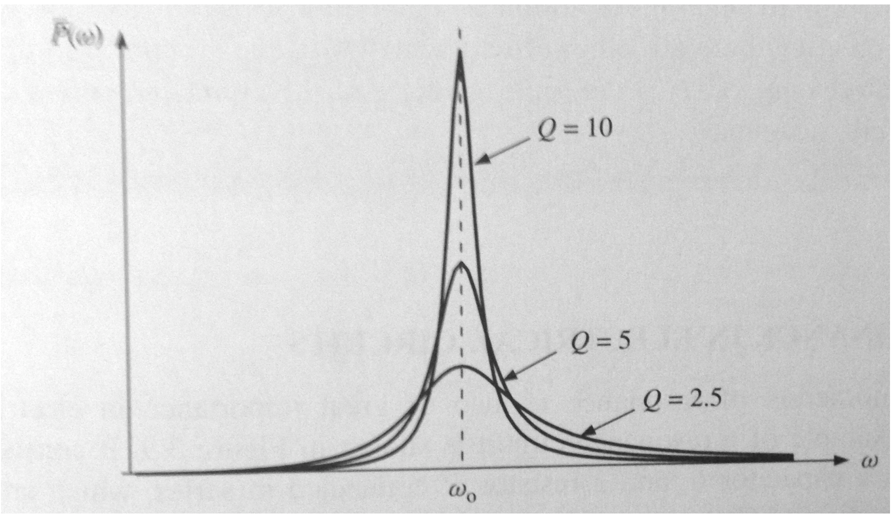
$$\begin{aligned}
P &\propto \omega A(\omega)\left\langle\cos\omega t\sin(\omega t - \delta)\right\rangle \\
&\propto \omega A(\omega)\sin\alpha \\
&= \frac{\gamma\omega^2 A^2(\omega)}{\omega_0^2 a} \\
&= \frac{\gamma\omega^2}{\omega_0^2 a} \times \frac{\left(a\omega_0^2\right)^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \\
&= \frac{a\gamma\omega^2\omega_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}
\end{aligned}$$

This has a maximum at the resonant frequency, and for higher *Q* values it is almost symmetric.

The width at half height is given by assuming the numerator to be constant, and consider the value at which the denominator doubles in size. That is

$$(\omega_0^2-\omega^2)^2+\omega^2\gamma^2 = 2\omega_0^2\gamma^2$$
$$\simeq 4\omega_0^2(\omega_0-\omega)^2+\omega_0^2\gamma^2$$
$$\omega_0-\omega = \frac{1}{2}\gamma$$
$$2(\omega_0-\omega)=\gamma$$
$$\frac{2(\omega_0-\omega)}{\omega_0}=\frac{\gamma}{\omega_0}=Q$$





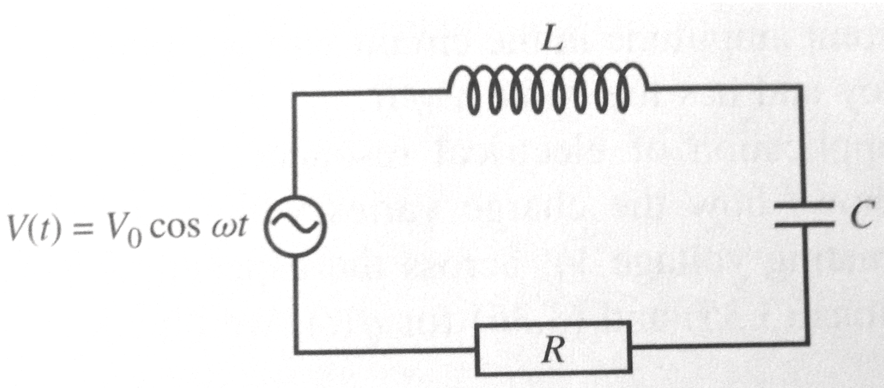
Example of two tuning forks. Note that tuning forks are high-Q systems, so the power resonance curve is very tight. Thus only forces of the same frequency will excite one.

Tap one and then touch it, and hear that we have excited the other one.

Change the frequency slightly and note that it can't be excited.

LRC circuits

Picture of circuit with alternating voltage.



From before we had

$$L\dot{I} + RI + Q/C = 0$$
$$\Rightarrow L\ddot{Q} + R\dot{Q} + Q/C = 0$$

Now we add a voltage

$$L\ddot{Q} + R\dot{Q} + Q/C = V_0 \cos\omega t$$
$$\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{LC}Q = \frac{V_0}{L}\cos\omega t$$
$$\ddot{Q} + \gamma\dot{Q} + \omega_0^2 Q = \frac{V_0}{L}\cos\omega t$$

This is the same as we have already been discussing. We have therefore

$$Q = Q_0(\omega)\cos(\omega t - \delta)$$
$$Q(\omega) = \frac{V_0}{\omega\sqrt{(1/\omega C - \omega L)^2 + R^2}}$$
$$I = \dot{Q} = -\omega Q_0(\omega)\sin(\omega t - \delta)$$
$$= \frac{V_0 \sin(\omega t - \delta)}{\sqrt{(1/\omega C - \omega L)^2 + R^2}}$$

You can develop this for power, and you will see there is an amplification effect. This is particularly useful if you can tune components, and that enables you to do things with signals such as radio waves.

Summary

1. We have started from a simple harmonic oscillator in different scenarios, and found the common features
2. We added damping following by forcing – you need damping if you have forcing

3. We have looked at two types of frequency: one that is the natural frequency of an oscillator, and one that is forced by the user. For a forced oscillator, we see the effects of the natural frequency in the transient states
4. The natural frequency also determines resonance in driven systems
5. The damping plays a role in stretching this resonance.

Next we will move on to talk about coupled oscillators.