Topic 3: Forced oscillators

Recap

What have we achieved so far…

1. Differential equation for simple harmonic oscillator

$$F = -kx = ma = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\ddot{x}^2$$

$$\ddot{x}^2 + \frac{k}{m}x = 0$$

$$x = x_0 \exp(i\omega t)$$

$$-\omega^2 x_0 \exp(i\omega t) + \frac{k}{m} x_0 \exp(i\omega t) = 0$$

$$\Rightarrow \omega = \sqrt{\frac{k}{m}}$$

2. Realised for a number of cases: Spring, pendulum, floating object, electric circuits, where we had alternative descriptions of the prefactors in the two terms.
3. Found that the simple harmonic oscillator conserves energy, which means there is no energy loss. This is not realistic, because we know that there are drains on energy, such as air or internal resistance to motion. We added a velocity term to create a general function

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

Tried general solution

$$x = A_0 \exp(i\beta t)$$
$$\dot{x} = i\beta A_0 \exp(i\beta t)$$
$$\ddot{x} = -\beta^2 A_0 \exp(i\beta t)$$

which leads to

$$x = A_0 \exp(i\beta t)$$
$$\dot{x} = i\beta A_0 \exp(i\beta t)$$
$$\ddot{x} = -\beta^2 A_0 \exp(i\beta t)$$
$$\Rightarrow -\beta^2 A_0 \exp(i\beta t) + i\gamma\beta A_0 \exp(i\beta t) + \omega_0^2 A_0 \exp(i\beta t) = 0$$
$$\Rightarrow \beta^2 - i\gamma\beta - \omega_0^2 = 0$$
$$\beta = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2} = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4}$$

Three cases …

Case 1

$$\omega^2 = \omega_0^2 - \gamma^2/4 > 0$$
$$x = A_0 \exp(i\beta t) = A_0 \exp(-\gamma t/2)\exp(i\omega t)$$

Case 2

$$-\alpha^2 = \omega_0^2 - \gamma^2/4 < 0$$
$$x = A_0 \exp(i\beta t)$$
$$\beta = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2} = \frac{i\gamma}{2} \pm i\alpha$$
$$x = \exp(-\gamma t/2)\left(A\exp(i\alpha t) + B\exp(-i\alpha t)\right)$$

Case 3

$$\omega_0^2 - \gamma^2/4 = 0$$
$$x = A_0 \exp(i\beta t)$$
$$\beta = \frac{i\gamma}{2}$$
$$x = \exp(-\gamma t/2)\left(A + Bt\right)$$

In these latter cases, we have two constants which we can obtain by boundary conditions, eg the values of x and its time derivative at $t = 0$.

What I have done is show how working with complex equations can make analysis rather simpler and appear less contrived.

In what follows I will first work with real variables first, then repeat with complex, and from then on stick with complex.

Recall that we introduced the figure of merit Q:

$$Q = \frac{\omega_0}{\gamma}$$
$$\left.\frac{\Delta E}{E}\right|_{\text{Period}} = -\frac{2\pi}{Q}$$

This was the topic of King's book chapter 2.

Today and next week we will look at Chapter 3.

Forced oscillator without damping

First case, a spring on an oscillator

Picture of spring

We apply a force of the form

$F = F_0 \cos\omega t$

This is added to the spring force to give a new equation

$$
\begin{aligned}
m\ddot{x} &= -kx + F_0 \cos\omega t \\
&= -kx + ka\cos\omega t \quad ; \quad a = F_0 / k
\end{aligned}
$$

$$\ddot{x} + \omega_0^2 x = \omega_0^2 a \cos\omega t$$

Note that we choose the angular frequency; unlike the case of the simple harmonic oscillator it is not a result of an equation.

The other difference is that the amplitude before was a boundary condition. Here we will derive it as an exact result, because it depends on the force. However, we will also find that it depends on frequency.

We will also have to explicitly consider a phase angle, which again will be determined by the equations and not simply a boundary condition.

Demonstration of a simple pendulum with different speeds of motion, showing change of phase.

So we assume the displacement can be written as

$$x = A(\omega)\cos(\omega t - \delta)$$

Note two things

1. The frequency is the same as the applied frequency. Intuitive, reflecting that the system has to do what is imposed on it
2. The phase angle means that the motion can lag in time, even to the point of being exactly out of phase. Not intuitive, but it follows.

We put this into our equation

$$
\begin{aligned}
&\ddot{x} + \omega_0^2 x = \omega_0^2 a \cos\omega t \\
&-\omega^2 A(\omega)\cos(\omega t - \delta) + \omega_0^2 A(\omega)\cos(\omega t - \delta) = \omega_0^2 a \cos\omega t \\
&A(\omega)(\omega_0^2 - \omega^2)\cos(\omega t - \delta)) = \omega_0^2 a \cos\omega t
\end{aligned}
$$

We now use the trigonometric identity

$$
\begin{aligned}
\cos(A - B) &= \cos A \cos B + \sin A \sin B \\
\cos(\omega t - \delta) &= \cos\omega t \cos\delta + \sin\omega t \sin\delta
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
A(\omega)(\omega_0^2 - \omega^2)\cos(\omega t - \delta)) &= \omega_0^2 a \cos\omega t \\
A(\omega)(\omega_0^2 - \omega^2)\left(\cos\omega t \cos\delta + \sin\omega t \sin\delta\right) &= \omega_0^2 a \cos\omega t
\end{aligned}
$$

Equation the cosine and sine terms gives the pair of equations

$$A(\omega)(\omega_0^2 - \omega^2)\cos\delta = \omega_0^2 a$$
$$A(\omega)(\omega_0^2 - \omega^2)\sin\delta = 0$$

The latter equation suggests that $\delta = 0$ or $\delta = \pi/2$.

Note that we have no freedom; boundary conditions do not come into the picture in the steady state.

When $\delta = 0$,

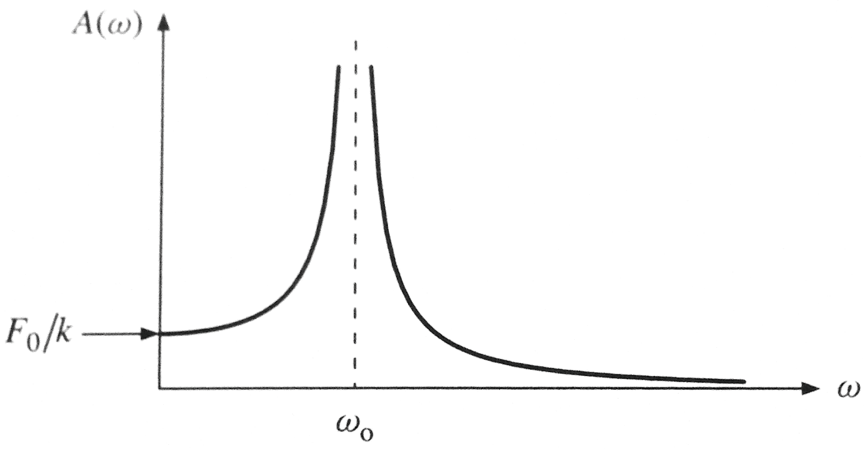
$$A(\omega) = \frac{\omega_0^2 a}{\omega_0^2 - \omega^2}$$

When $\delta = \pi/2$

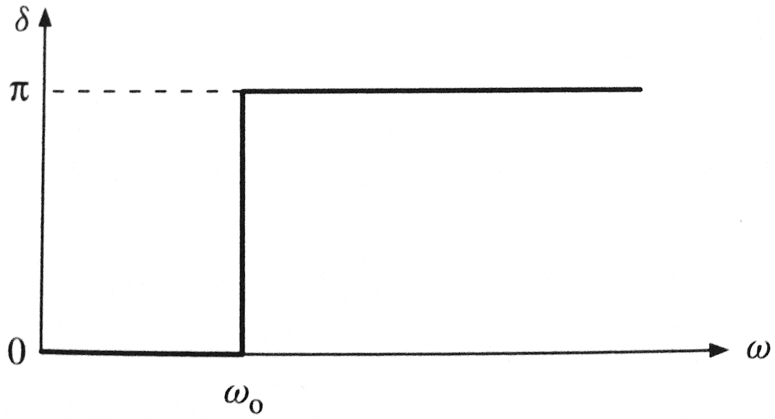
$$A(\omega) = -\frac{\omega_0^2 a}{\omega_0^2 - \omega^2}$$

For a positive amplitude, we therefore expect that the first solution works for the case $\omega < \omega_0$, and the second case for $\omega > \omega_0$. The phase switches at $\omega = \omega_0$.

Show picture from King (p 53) and note the limiting cases



(a)



Note that at low frequency,

$$A(\omega) = \frac{\omega_0^2 a}{\omega_0^2 - \omega^2} = a = F_0 / k$$

At high frequency

$$A(\omega) = -\frac{\omega_0^2 a}{\omega_0^2 - \omega^2} = \frac{\omega_0^2 a}{\omega^2 - \omega_0^2} \to 0$$

But the key thing is that when the frequency of the applied force equals the natural frequency of the oscillator, we get an infinite amplitude. This is called "resonance". It suggests that all the energy pumped into the system from the external force is converted to the energy of the oscillator

Examples:

1. The Millennium bridge
2. My old Allegro car
3. Washing machine
4. Child's swing
5. Carrying bowl of water
6. Walking
7. Wine glass
8. Seat on driver's bus

Now let's repeat this using complex representation

$$x = A(\omega)\exp i(\omega t - \delta)$$

$$\ddot{x} + \omega_0^2 x = \omega_0^2 a \exp i\omega t$$
$$-\omega^2 A(\omega)\exp i(\omega t - \delta) + \omega_0^2 A(\omega)\exp i(\omega t - \delta) = \omega_0^2 a \exp i\omega t$$
$$A(\omega)(\omega_0^2 - \omega^2)\exp i(\omega t - \delta)) = \omega_0^2 a \exp i\omega t$$
$$A(\omega)(\omega_0^2 - \omega^2)\exp(-i\delta) = \omega_0^2 a$$

which only works if the complex exponential term has a real value, which must be either +1 or –1.

The big problem with this model is that infinite amplitude, which can only be tackled if we allow energy leak.

Forced oscillations with damping

King does this with real quantities, but I will here use the complex notation.

The master equation is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \omega_0^2 a \exp(i\omega t)$$
$$x = A(\omega)\exp\big(i(\omega t - \delta)\big)$$
$$\dot{x} = i\omega A(\omega)\exp\big(i(\omega t - \delta)\big)$$
$$\ddot{x} = -\omega^2 A(\omega)\exp\big(i(\omega t - \delta)\big)$$
$$-\omega^2 A(\omega)\exp\big(i(\omega t - \delta)\big) + i\gamma\omega A(\omega)\exp\big(i(\omega t - \delta)\big) + \omega_0^2 A(\omega)\exp\big(i(\omega t - \delta)\big) = \omega_0^2 a \exp(i\omega t)$$
$$A(\omega)\exp(-i\delta)\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) = \omega_0^2 a$$

Take the real and imaginary parts separately:

$$A(\omega)\exp(-i\delta)\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) = \omega_0^2 a$$
$$A(\omega)\left(-\omega^2 + i\gamma\omega + \omega_0^2\right) = \omega_0^2 a\left(\cos\delta + i\sin\delta\right)$$
$$A(\omega)\left(\omega_0^2 - \omega^2\right) = \omega_0^2 a\cos\delta$$
$$A(\omega)\gamma\omega = \omega_0^2 a\sin\delta$$

Dividing the last two gives

$$\frac{A(\omega)\gamma\omega}{A(\omega)\left(\omega_0^2 - \omega^2\right)} = \frac{\omega_0^2 a\sin\delta}{\omega_0^2 a\cos\delta}$$
$$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

Adding the squares gives

$$\left(A(\omega)\left(\omega_0^2 - \omega^2\right)\right)^2 + \left(A(\omega)\gamma\omega\right)^2 = \left(\omega_0^2 a\right)^2\left(\cos^2\delta + \sin^2\delta\right) = \left(\omega_0^2 a\right)^2$$
$$A(\omega)\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)^{1/2} = \omega_0^2 a$$
$$A(\omega) = \frac{\omega_0^2 a}{\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)^{1/2}}$$

So we come to the “so what?” question. What does all this mean?

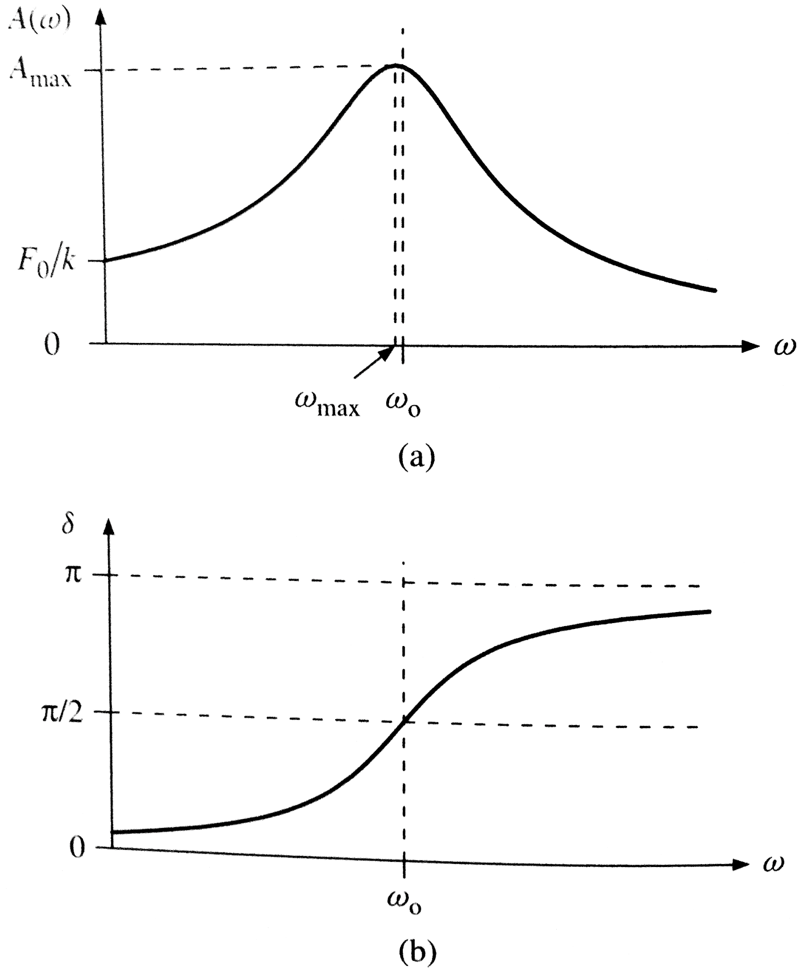
Take some limits:

$$\omega \to 0, \quad \tan\delta \to 0 \quad \Rightarrow \quad \delta \to 0$$
$$\omega \to \infty, \quad \tan\delta \to -0 \quad \Rightarrow \quad \delta \to \pi$$
$$\omega = \omega_0, \quad \tan\delta = \infty \quad \Rightarrow \quad \delta = \pi/2$$

and

$$\omega \to 0, \quad A(\omega) \to a = F_0/k$$
$$\omega \to \infty, \quad A(\omega) \to 0$$
$$\omega = \omega_0, \quad A(\omega) = a\omega_0/\gamma$$

Look at plots from King p 56.



Let's work out the frequency at which the response is at a maximum – note that when we have no damping this is the natural frequency.

$$A(\omega) = \frac{\omega_0^2 a}{\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)^{1/2}}$$

$$\frac{\mathrm{d}A}{\mathrm{d}\omega} = -\frac{1}{2}\frac{\omega_0^2 a}{\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)^{3/2}}\left(-4\omega\left(\omega_0^2 - \omega^2\right) + 2\gamma^2\omega\right)$$

$$= 0 \text{ if}$$

$$2\left(\omega_0^2 - \omega_{max}^2\right) + \gamma^2 = 0$$

$$\omega_{max}^2 = \omega_0^2 - \gamma^2/2$$

which is slightly further away from the natural frequency for the underdamped oscillator.

At this frequency, we have

$$
\begin{aligned}
A_{\max} &= \frac{\omega_0^2 a}{\left(\left(\omega_0^2-\left(\omega_0^2-\gamma^2/2\right)\right)^2+\gamma^2\left(\omega_0^2-\gamma^2/2\right)\right)^{1/2}} \\
&= \frac{\omega_0^2 a}{\left(\gamma^4/4+\gamma^2\omega_0^2-\gamma^4/2\right)^{1/2}} \\
&= \frac{\omega_0^2 a}{\left(\gamma^2\omega_0^2-\gamma^4/4\right)^{1/2}} \\
&= \frac{\omega_0 a/\gamma}{\left(1-\gamma^2/4\omega_0^2\right)^{1/2}}
\end{aligned}
$$

Recall our definition of Q, which defines the extent of damping and relates to energy loss

$$
Q = \frac{\omega_0}{\gamma}
$$

$$
\left.\frac{\Delta E}{E}\right|_{\text{Period}} = -\frac{2\pi}{Q}
$$

So we can rewrite the equations in terms of Q to give us a good idea as to what to expect, given that we have a bit of a feel for Q:

$$
A_{\max} = \frac{\omega_0 a/\gamma}{\left(1-\gamma^2/4\omega_0^2\right)^{1/2}} = \frac{aQ}{\left(1-1/4Q^2\right)^{1/2}}
$$

$$
\omega_{\max} = \left(\omega_0^2-\gamma^2/2\right)^{1/2} = \omega_0\left(1-1/2Q^2\right)^{1/2}
$$

When Q is large, ie small damping, the results tend towards

$$
\omega_{\max} \to \omega_0
$$

$$
A_{\max} \to aQ
$$

which means that Q defines the extent of the amplitude. In effect, the oscillator is an amplifier with Q as the amplification factor.

Note two things as Q gets larger, linking to case of no damping before:

1. The phase changes only in the vicinity of the resonant frequency
2. The amplitude increases more
3. The peak frequency becomes closer to the resonant frequency.

Look at what happens when $Q = 5$, which is a small value:

$$
A_{\max} = \frac{\omega_0 a/\gamma}{\left(1-\gamma^2/4\omega_0^2\right)^{1/2}} = \frac{aQ}{\left(1-1/4Q^2\right)^{1/2}}
$$

$$
\frac{\omega_{\max}}{\omega_0} = \sqrt{1-1/2Q^2} = \sqrt{1-1/50} \simeq 0.99
$$

Example of pendulum

Draw pendulum with horizontal displacement. From before we had the following and can extend this for damping

$$m\ddot{x} + \frac{mg}{L}x = 0$$

$$\omega_0^2 = \frac{g}{L}$$

$$m\ddot{x} + m\omega_0^2 x = 0$$

$$\ddot{x} + \gamma x + \omega_0^2 x = 0$$

But now we need to use not *x* but *X*, which is the actual displacement.

$$X = x + \eta_0 \cos\omega t$$

$$\ddot{x} + \gamma x + \omega_0^2 x = \ddot{X} + \gamma \dot{X} + \omega_0^2 (X - \eta_0 \cos\omega t) = 0$$

$$\ddot{X} + \gamma \dot{X} + \omega_0^2 X = \omega_0^2 \eta_0 \cos\omega t$$

Show pendulum moving

1. Very slowly moving, showing phase is zero
2. Very fast, no motion at all
3. Fast but not so fast, show out of phase
4. Try to hit resonance