Topic 2: Damped harmonic motion

Introduction

So far in the discussion there is nothing in our models or equations to remove energy from the oscillations. We might imagine though that there are things that could remove energy. Examples are

1. Friction in any moving parts (including the spring)
2. Air resistance

We might guess that with some resistance to motion the displacement will dampen with time, and perhaps we might have a solution of the form

$$x(t) = x_0 \exp(-\beta t)\cos(\omega t)$$

Air resistance and friction do not exert a force when an object is stationary, and so our sensible guess is that they had a force that depends on velocity. Specifically we can write

$$F = -b\dot{x}$$

(highlight the minus sign – the force acts in the opposite direction to the velocity)

On this basis we might rewrite our differential equation for motion as

$$m\ddot{x} = -kx - b\dot{x}$$

which is a small tweak of what we were looking at in the last lecture.

We can rewrite this as

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

where we have defined two variables (one of which follows exactly from the definitions from lecture 1).

Solution without complex numbers

For the moment, we will work without using complex numbers. This solution may appear a bit contrived, but it isn't. It will work rather more naturally when we use complex numbers.

We assume a solution to the differential equation for a damped oscillator of the form

$$x(t) = \exp(-\beta t) f(t)$$

We now compute the differentials:

$$\dot{x}(t) = -\beta\exp(-\beta t)f(t) + \exp(-\beta t)\dot{f}(t) = \exp(-\beta t)\left(-\beta f(t) + \dot{f}(t)\right)$$
$$\ddot{x}(t) = -\beta\exp(-\beta t)\left(-\beta f(t) + \dot{f}(t)\right) + \exp(-\beta t)\left(-\beta\dot{f}(t) + \ddot{f}(t)\right) = \exp(-\beta t)\left(\beta^2 f(t) - 2\beta\dot{f}(t) + \ddot{f}(t)\right)$$

Inserting into our differential equation gives

$$\begin{aligned}\ddot{x} + \gamma\dot{x} + \omega_0^2 x &= \exp(-\beta t)\left(\beta^2 f(t) - 2\beta\dot{f}(t) + \ddot{f}(t) - \beta\gamma f(t) + \gamma\dot{f}(t) + \omega_0^2 f(t)\right) \\ &= \exp(-\beta t)\left(\ddot{f}(t) + (\gamma - 2\beta)\dot{f}(t) + \left(\beta^2 - \beta\gamma + \omega_0^2\right)f(t)\right) \\ &= 0\end{aligned}$$

Given that we invented the variable β, we can set it to be whatever we want (this is where things look a bit contrived, but don't worry!). So let's set it as $\beta = \gamma/2$. Let's proceed by writing out the equation for $f(t)$:

$$\begin{aligned}\ddot{f}(t) + (\gamma - 2\beta)\dot{f}(t) + \left(\beta^2 - \beta\gamma + \omega_0^2\right)f(t) &= \ddot{f}(t) + \left(\gamma^2/4 - \gamma^2/2 + \omega_0^2\right)f(t) \\ &= \ddot{f}(t) + \left(\omega_0^2 - \gamma^2/4\right)f(t) \\ &= 0\end{aligned}$$

This defines three regimes, depending on whether the prefactor of $f(t)$ is positive, negative or zero.

Underdamped regime: $\omega_0^2 > \gamma^2/4$

Here we can write the equation

$$\begin{aligned}&\ddot{f}(t) + \omega^2 f(t) = 0 \\ &\qquad f(t) = A_0\cos(\omega t) \\ &\qquad x(t) = A_0\exp(-\gamma t/2)\cos(\omega t)\end{aligned}$$

Note that this defines a damped vibration of angular frequency $\omega < \omega_0$.

Overdamped regime: $\omega_0^2 < \gamma^2/4$

Here we can write the equation

$$\begin{aligned}&\ddot{f}(t) - \alpha^2 f(t) = 0 \\ &\qquad f(t) = A\exp(\alpha t) + B\exp(-\alpha t) \\ &\qquad x(t) = \exp(-\gamma t/2)\left(A\exp(\alpha t) + B\exp(-\alpha t)\right)\end{aligned}$$

The result here is a slowly decaying function with no oscillation.

Note that the coefficients A and B depend on the starting conditions, for example the initial position and velocity, which are called the "boundary conditions".

Consider the case at t = 0, giving

$$x(t) = \exp(-\gamma t/2)\big(A\exp(\alpha t) + B\exp(-\alpha t)\big) \Rightarrow x(0) = A + B$$

$$\dot{x}(t) = -\frac{\gamma}{2}x(t) + \exp(-\gamma t/2)\big(\alpha A\exp(\alpha t) - \alpha B\exp(-\alpha t)\big) \Rightarrow \dot{x}(0) = -\frac{\gamma}{2}(A+B) + \alpha(A-B)$$

So let's take the case where we have an initial displacement held at zero velocity before release. Thus we have

$$\begin{aligned}
&x(0) = A + B \\
&\quad 0 = -\frac{\gamma}{2}(A+B) + \alpha(A-B) \\
&\quad \Rightarrow \frac{\gamma}{2}(A+B) = \alpha(A-B) \\
&\quad \Rightarrow \left(\alpha - \frac{\gamma}{2}\right)A = \left(\alpha + \frac{\gamma}{2}\right)B \\
&\quad A = x(0) - B
\end{aligned}$$

You can see that these equations will allow us to specify values for *A* and *B*.

Critical damping: $\omega_0^2 = \gamma^2/4$

Here the differential equation is much simpler because the second term now vanishes:

$$\ddot{f}(t) = 0$$

This has solution

$$\begin{aligned}
&f(t) = A + Bt \\
&x(t) = \exp(-\gamma t/2)\big(A + Bt\big)
\end{aligned}$$

Again, the two parameters *A* and *B* can be obtained from boundary conditions.

Rate of energy loss

We write the energy as a sum of kinetic and potential energies:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

We consider here the underdamped oscillator, and substitute in our general solution:

$$\begin{aligned}
\dot{x} &= -\frac{\gamma}{2}A_0\exp(-\gamma t/2)\cos(\omega t) - A_0\exp(-\gamma t/2)\omega\sin(\omega t) \\
&= -A_0\omega\exp(-\gamma t/2)\left(\frac{\gamma}{2\omega}\cos(\omega t) + \sin(\omega t)\right) \\
&\approx -A_0\omega\exp(-\gamma t/2)\sin(\omega t)
\end{aligned}$$

where we have assumed that the prefactor on the cosine term is much less than 1 in value so that we can ignore this term.

This we can write the energy as

$$
\begin{aligned}
E &= \frac{1}{2} m A_0^2 \omega^2 \exp(-\gamma t)\sin^2(\omega t) + \frac{1}{2} k A_0^2 \exp(-\gamma t)\cos^2(\omega t) \\
&= \frac{1}{2} A_0^2 \exp(-\gamma t)\left(m\omega^2 \sin^2(\omega t) + k\cos^2(\omega t)\right)
\end{aligned}
$$

So assuming damping is light we can write

$$\omega^2 \simeq \omega_0^2 = k / m$$

and hence we have

$$
\begin{aligned}
E &= \frac{1}{2} A_0^2 \exp(-\gamma t)\left(k\sin^2(\omega t) + k\cos^2(\omega t)\right) \\
&= \frac{1}{2} k A_0^2 \exp(-\gamma t) \\
&= E_0 \exp(-\gamma t)
\end{aligned}
$$

The quantity $\tau = 1/\gamma$ defines the decay time of the energy in the oscillation.

Quality factor

It is common to define a figure of merit to describe the damping. This is a dimensionless number given the name "Quality Factor" and the symbol *Q*. It's practical meaning will become apparent through the discussion here and in subsequent lectures.

The quality factor is defined as

$$Q = \frac{\omega_0}{\gamma}$$

This clearly has no dimensions.

We can use this to look at the angular frequency of the underdamped oscillator:

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \omega_0^2\left(1 - \frac{1}{4Q^2}\right)$$

We take two examples

$$
\begin{aligned}
&Q = 10 \Rightarrow \omega^2 = \omega_0^2\left(1 - \frac{1}{400}\right) \quad ; \quad \frac{\omega_0 - \omega}{\omega_0} \approx \frac{1}{8}\% \\
&Q = 2 \Rightarrow \omega^2 = \omega_0^2\left(1 - \frac{1}{16}\right) \quad ; \quad \frac{\omega_0 - \omega}{\omega_0} \approx 3\%
\end{aligned}
$$

Clearly even quite a low value of the quality factor doesn't lead to a large change in the angular frequency (which justifies our discussion concerning the energy loss before).

Now we consider the energy loss

$$\dot{E} = -\gamma E$$
$$\Delta E = \dot{E}\Delta t = -\gamma E \Delta t$$
$$\frac{\Delta E}{E} = -\gamma \Delta t$$

If we consider a time step of one period of oscillation, we have

$$\Delta t = \frac{2\pi}{\omega}$$
$$\frac{\Delta E}{E} = -\gamma \frac{2\pi}{\omega} = -\frac{2\pi}{Q}$$

Thus Q defines also the relative energy loss per cycle of oscillation.

LCR circuit

Consider now an electrical circuit consisting of a resistor, and inductor and a capacitor in series.

Kirchoff's law gives

$$L\dot{I} + RI + \frac{q}{C} = 0$$
$$\Rightarrow L\ddot{q} + R\dot{q} + \frac{q}{C} = 0$$
$$\Rightarrow \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0$$
$$\gamma = \frac{R}{L} \quad ; \quad \omega_0^2 = \frac{1}{LC}$$
$$\Rightarrow \ddot{q} + \gamma \dot{q} + \omega_0^2 q = 0$$

Everything follows from before, namely we have a damped oscillator in terms of the charge rather than a mechanical displacement. We also have a quality factor:

$$Q = \frac{\omega_0}{\gamma} = \frac{1}{\sqrt{LC}}\frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Damped oscillator using complex numbers

Here we try using a simple complex oscillator as the trial function.

$$x(t) = x_0 \exp(i\beta t)$$
$$\dot{x}(t) = x_0 i\beta \exp(i\beta t)$$
$$\ddot{x}(t) = -x_0 \beta^2 \exp(i\beta t)$$

Substituting into the differential equation gives

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$
$$\Rightarrow -x_0 \beta^2 \exp(i\beta t) + i x_0 \gamma\beta \exp(i\beta t) + \omega_0^2 x_0 \exp(i\beta t) = 0$$
$$\Rightarrow -\beta^2 + i\gamma\beta + \omega_0^2 = 0$$
$$\Rightarrow \beta^2 - i\gamma\beta - \omega_0^2 = 0$$
$$\Rightarrow \beta = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2} = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2 / 4}$$

For the case where $\omega_0^2 > \gamma^2 / 4$, ie underdamped, the square root is a real quantity, and we have the motion

$$x(t) = x_0 \exp(i\beta t) = x_0 \exp(-\gamma t / 2) \exp\left(i\sqrt{\omega_0^2 - \gamma^2 / 4}t\right) = x_0 \exp(-\gamma t / 2) \exp(i\omega t)$$

Taking the real part gives us exactly as before.

For the case where $\omega_0^2 < \gamma^2 / 4$, ie underdamped, the square root is an imaginary quantity, and we have the motion

$$x(t) = x_0 \exp(i\beta t) = x_0 \exp(-\gamma t / 2) \exp\left(\pm\sqrt{\gamma^2 / 4 - \omega_0^2}\,t\right)$$

Taking both positive and negative solutions gives the solutions we obtained before.

Summary

1. We have looked at extending the equation of motion by including a damping term that depends on the velocity (or first time derivative of the oscillating quantity).
2. The equations have been cast in a form that allows us to define regimes of underdamped, overdamped and critically damped.
3. Underdamping leads to an oscillatory solution with damping.
4. Overdamping leads to a slow relaxation of the variable back to its equilibrium value.
5. Critical damping allows a faster relaxation of the variable back to its equilibrium value.
6. We have obtained an equation for the rate of energy loss of an underdamped oscillator.
7. We have defined a quality factor that characterises the way that the energy is lost, or the amount that the angular frequency is shifted due to damping.