Topic 10: Summary

Wave transmission and boundaries

Consider a string that has a discontinuous change in properties at some point, which we call x = 0.

To the left, we have mass per unit length of μ_1 and velocity of wave c_1, and to the right we have μ_2 and c_2.

We have an incident wave travelling from left to right. Part of this wave is transmitted, and part is reflected. The boundary condition is that the displacement of both parts at x = 0 is identical. You also need to have a continuous derivative or else you form a kink. Hence we have

$$y_1(0) = y_2(0) \quad ; \quad \left.\frac{\partial y_1}{\partial x}\right|_{x=0} = \left.\frac{\partial y_2}{\partial x}\right|_{x=0}$$

Define the three waves, noting the sign of k defines the direction of the wave

$$y_i = A_i \sin\left(k_1 x - \omega_1 t\right)$$
$$y_r = A_r \sin\left(-k_1 x - \omega_1 t\right)$$
$$y_t = A_t \sin\left(k_2 x - \omega_1 t\right)$$

Note that the frequency of the transmitted wave stays the same, but the wavelength changes, because the junction needs to move by both waves at the same frequency.

We have boundary conditions:

$$y_1(0) = y_2(0) \quad \Rightarrow \quad A_i + A_r = A_t$$
$$\left.\frac{\partial y_1}{\partial x}\right|_{x=0} = \left.\frac{\partial y_2}{\partial x}\right|_{x=0} \quad \Rightarrow \quad -A_i k_1 + A_r k_1 = -A_t k_2$$

Too many unknowns as it stands, but we have the ratio of wave vectors from our given facts:

$$\omega = ck \quad \Rightarrow \quad c_1 k_1 = c_2 k_2 \quad \Rightarrow \quad k_1 / k_2 = c_2 / c_1$$

Hence we have

$$\left(A_i - A_r\right) c_2 = A_t c_1$$

We can combine these two equations:

$$A_i + A_r = A_t$$

$$A_i - A_r = A_t \frac{c_1}{c_2}$$

$$2A_i = A_t \left(1 + \frac{c_1}{c_2}\right) \quad \Rightarrow \quad A_t = \frac{2c_2}{c_1 + c_2} A_i$$

$$\frac{c_2}{c_1}\left(A_i - A_r\right) = A_t$$

$$\frac{c_2}{c_1}\left(A_i - A_r\right) - \left(A_i + A_r\right) = 0$$

$$c_2\left(A_i - A_r\right) = c_1\left(A_i + A_r\right) \quad \Rightarrow \quad \left(c_1 + c_2\right)A_r = \left(c_2 - c_1\right)A_i \quad \Rightarrow \quad A_r = \frac{c_2 - c_1}{c_1 + c_2} A_i$$

Example of a fixed end, such that $c_2 = 0$

$$A_t = \frac{2c_2}{c_1 + c_2} A_i = 0$$

$$A_r = \frac{c_2 - c_1}{c_1 + c_2} A_i = \frac{-c_1}{c_1} A_i = -A_i$$

This means the reflected wave goes back with opposite displacement. A peak hits the wall and reflects as a trough. This is the standard reflection condition.

Now take the case that $c_1 = c_2$. It follows that

$$A_t = \frac{2c_2}{c_1 + c_2} A_i = A_i$$

$$A_r = \frac{c_2 - c_1}{c_1 + c_2} A_i = 0$$

Air in a column

Very quickly to note, given that all this has links with musical instruments, that we can think quickly about air in a pipe. We could have open or closed ends.

At an open end, the pressure is zero, so any wave set up in the air will have a maximum at the open end.

At the closed end, nothing moves, so you will have a node.

Thus a pipe will support different harmonics depending on whether the ends are open or closed. Typically one end is open and one closed.

Musical instruments

Here the idea is to exploit forced oscillations, so that you can convert the energy of the primary oscillator (eg a string) into a sound wave that propagates through air. Typically the primary oscillator drives oscillations of another component, such as the body of a violin which in turn forces oscillations of the air to generate sound waves. We have the problem of resonant frequencies, but complex three-dimensional (or two-dimensional) objects will have many resonances, and we recall that resonances can be broad with some damping, so that the resonator will pick up a wide range of frequencies. But note that there will inevitably be an art in designing these resonators and a critical choice of materials.

Note also that musical notes will die away. Better instruments will have a reasonable degree of sustain, ie relative high Q factor. But you can add a light bit of friction, mechanically or by hand, to lower Q to have sharper cut-off of the note.

Harmonics

A string will resonate with some fixed frequencies. We have previously seen the way that you get harmonics. If we have harmonics of different intensity, we can combine all the harmonics to give a Fourier series. This will be an addition of sine waves.

Note that a pure sine wave has a very boring sound.

Plot a graph of intensity vs frequency with delta functions. Depending on the intensities you can get different wave forms, for example square wave and sawtooth.

The square wave – presumably you have done this – is made up of harmonics $n = 1, 3, 5$ etc, with ratios of intensity of $1/n$.

A sawtooth wave has all harmonics with weight 1/n but with alternating sign.

A triangular wave has again only odd harmonics, with weighting $1/n^2$ and alternate signs.

We also have two other factors

Non-infinite Q

This gives a decay of the signal, but more than that, the Fourier transform gives peaks that are not delta functions. This means that we bring in additional waves around the frequency of the harmonics, that can make the sound richer and more interesting.

Force oscillation

There is a sense in which sound in musical instruments is forced, for example pulling a bow on a violin, or blowing in a wind instrument. Recall that in a forced oscillator, we also have transients at the start, which can make the sound more interesting.

These things are exploited in digital music.

Videos

Show videos of

1. Cymbal, noting that in 2d it is hard to get a pure sound, so we get many harmonics which may not match, but interesting
2. Metronomes moving from transient to steady state
3. Complex pendulum, making the point that you know how to solve for this even though it might be complex.

4. Large water wave, making the point that here the wave shows dispersion, namely that the wave doesn't have a simple linear relationship between frequency and wavelength. Water waves are interesting – but complicated – in that they depend on the depth, and the peak and troughs will move at different speeds, particularly as the depth gets shallower on reaching the shore.

Revision points: essential skills

1. We need for any example to establish the forces acting. The easiest thing is to set up an equation for the potential energy – whether it be for springs or gravitational potential energy – and compute the force as the negative of the derivative. Provided you remember the sign of the derivative, you will always get the signs of the forces correctly. Getting a consistent set of signs is the biggest challenge.
2. Note that this works equally well for a complex set of coupled oscillators as it does for a single oscillation. In principle you could solve the equations for the complex pendulum I showed you a video of.
3. For damping, there is an additional force that is proportional to the velocity, with a minus sign because it acts against the direction of motion. The point of this force is that it doesn't act when the object is at rest.
4. You need to then equate the force on the object to mass times acceleration.
5. For forced systems, you need to be add in the external force. Note that sometimes this is an explicit force, but often it is given in the form of an enforced periodic displacement. We have dealt with both systems.
6. Often we have to add up waves, using standard trigonometric relations. Recall that when you add two waves of similar frequency, you get a product of two terms, one being the average frequency and one being derived from the difference in frequencies. This leads to beating, which we demonstrated with the two tuning forks, and which is important in transient phenomena.
7. We have dealt with two types of wave equation. One where there is no spatial derivative (the fixed oscillators) and one where there is a second derivative with respect to distance. The latter gives travelling waves.
8. We have seen three derivations of this differential equation, both atomic and continuum. You will also see a derivation in electromagnetic theory, where you can find electromagnetic waves with a velocity of light given by fundamental parameters.
9. The solution is a wave where you get not only the frequency times time term but also a wave vector times position component of the argument of the sine wave.
10. You need to understand the concept of wave vector. It is not hard, but it is a common concept in Physics.
11. Be aware that although we like to think in terms of real numbers, many problems of waves are best tackled using complex number representations. We saw that using complex numbers gave a much easier solution to the damped oscillator, where we got

both under-damped and over-damped solutions from one equation, and where we didn't need to make any prior assumptions.

12. For complex cases, you need to work with matrices!