Topic 1: Simple harmonic motion

Introduction

Why do we need to know about waves

1. Ubiquitous in science – nature likes wave solutions to equations
2. They are an exemplar for some essential Physics skills: differential equations, solutions of eigenvector equations

Show a number of visual examples

1. Quartz clock
2. Cuckoo clock
3. Walking
4. Atoms vibrating in a crystal
5. Light
6. Neutron and electron scattering – wave/particle duality
7. Sound = music
8. Oscillating astronomical objects, http://www.aavso.org/lcotw/ae-ursae-majoris
9. Vibrating Earth, http://icb.u-bourgogne.fr/nano/MANAPI/saviot/terre/index.en.html
10. Water waves, http://vimeo.com/channels/63693/5301095

Where are we heading in this course

1. Simple harmonic motion
2. Damped harmonic motion
3. Forced harmonic motion
4. Coupled oscillators
5. Travelling waves

Books

Vibrations and Waves, GC King (Manchester, Wiley) and AP French (MIT, CBS). I will mostly follow King, but French is the more traditional book.

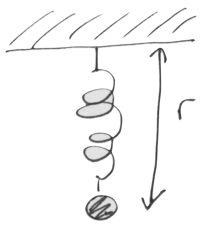
Loads of stuff on the web! Including MIT courseware videos.

Most simple oscillator

Simple model

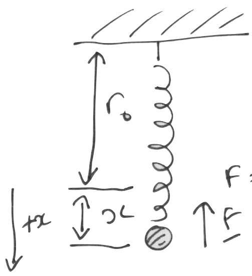
Start with mass and spring, assuming the spring has no mass.

Draw pictures showing the situation of rest, pull and compress.



$E = \frac{1}{2}k(r - r_0)^2$

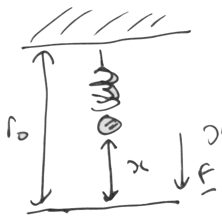
At $r = r_0$, $E = 0 \equiv$ equilibrium



$r = r_0 + x$

$F = -\frac{\partial E}{\partial r} = -k(r - r_0) = -kx$

$\therefore$ Force is upwards



x is $-$ve

$F = -kx =$ positive in down direction

Question: how do we define the energy?

We can write the energy as a Taylor series. In practice systems are designed to ensure that the first term is dominant. This works well with a spring because although it expands quite a bit, the motion on a single turn is very small.

Equations of motion

From harmonic energy calculate the spring force as minus the derivative. Show that you get exactly what you want.

$$E = \frac{1}{2}kx^2$$

$$F = -\frac{dE}{\mathrm{d}x} = -kx$$

$$\frac{\mathrm{d}^2E}{\mathrm{d}x^2} = k$$

We use force = mass × acceleration. Write master equation:

$$F = -kx = ma = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\ddot{x}^2$$

$$-kx = m\ddot{x}^2$$

$$m\ddot{x}^2 + kx = 0$$

Key thing here – the second differential is within a scale factor of the original function. Ask the audience if they know of any functions where this works.

First try the cosine or sine functions

$$x = A\cos\omega t$$

$$\dot{x} = -A\omega\sin\omega t$$

$$\ddot{x} = -A\omega^2\cos\omega t$$

$$m\ddot{x} + kx = -Am\omega^2\cos\omega t + kA\cos\omega t$$

$$= \left(-m\omega^2 + k\right)A\cos\omega t = 0$$

$$\omega^2 = k/m$$

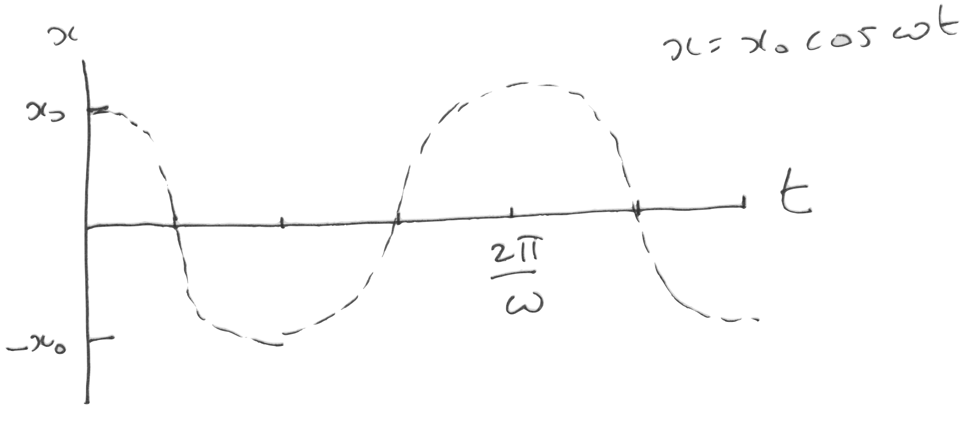
Note that this also works if we use sine instead of cosine.

Note dimension of units. k has units of energy/distance^2 = mass/time^2. Thus ω has units of 1/time, which is like a frequency. But it isn't a frequency alone, because the cosine repeats in units of 2π. So we have

$$\omega = 2\pi f = 2\pi/\tau$$

Frequency has units of Hz = 1/second. Angular frequency has units of radians per second.

Sketch figure of cosine, highlighting the period.



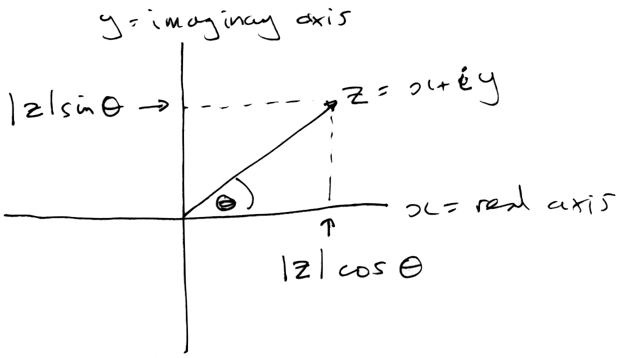
Highlight definition of angular frequency.

Note also that ω depends on k and m, but not at all on A. That means it is completely independent of the starting point, and hence of the energy within the vibration.

The other possible solution is exponential:

$$
\begin{aligned}
x &= A\exp\alpha t \\
\dot{x} &= A\alpha\exp\alpha t \\
\ddot{x} &= A\alpha^2\exp\alpha t \\
m\ddot{x} + kx &= Am\alpha^2\exp\alpha t + kA\exp\alpha t \\
&= \left(m\alpha^2 + k\right)A\exp\alpha t = 0 \\
\alpha^2 &= -k/m \\
\alpha &= i\sqrt{k/m} = i\omega \\
x &= A\exp i\omega t = A(\cos\omega t + i\sin\omega t) \\
\dot{x} &= i\omega A\exp i\omega t \\
\ddot{x} &= -\omega^2 A\exp i\omega t = -\omega^2 x \\
m\ddot{x} + kx &= -m\omega^2 x + kx = 0
\end{aligned}
$$

Display on *Argand diagram*. Note that we have a vector of fixed length rotating in the complex plane.

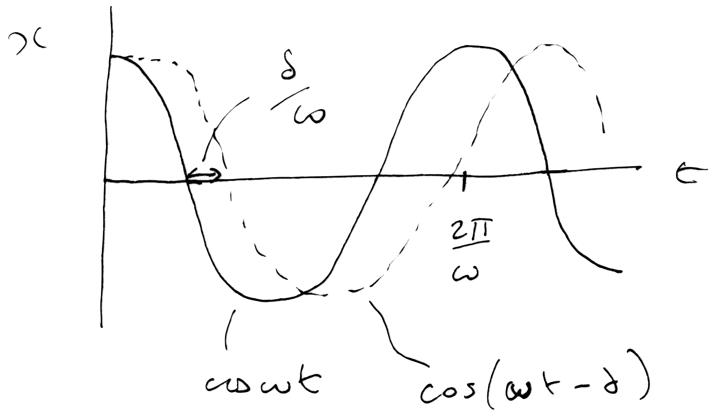


$$
\begin{aligned}
z &= |z|\left(\cos\theta + i\sin\theta\right) \\
&= |z|\,e^{i\theta}
\end{aligned}
$$

So can we use a combination of functions, such as

$$
\begin{aligned}
x &= A\cos\omega t + B\sin\omega t \\
&= A_0\left(a\cos\omega t + b\sin\omega t\right) \quad ; \quad a^2 + b^2 = 1 \\
&= A_0\left(\cos\delta\cos\omega t + \sin\delta\sin\omega t\right) \\
&= A_0\cos(\omega t - \delta)
\end{aligned}
$$

So we see that these sorts of combinations act as a phase shift.



The value of δ comes down to our definition of time, and particularly what we call the origin of time. However, when we release a ball on a spring, the sensible definition would have δ = 0, which defines what we call our boundary condition. In the case of releasing the ball from a standing start, we define both the phase shift (zero) and the amplitude (how far we extended the string).

Energy of the oscillator

$$\begin{aligned}
E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\
&= \frac{1}{2}m\left(\omega A\sin\omega t\right)^2 + \frac{1}{2}k\left(A\cos\omega t\right)^2 \\
&= \frac{1}{2}A^2\left(m\omega^2\sin^2\omega t + k\cos^2\omega t\right) \\
&= \frac{1}{2}A^2\left(k\sin^2\omega t + k\cos^2\omega t\right) \\
&= \frac{1}{2}A^2k
\end{aligned}$$

Note that there is no dependence on time, so we have shown that energy is conserved.

Now let's look at the average components

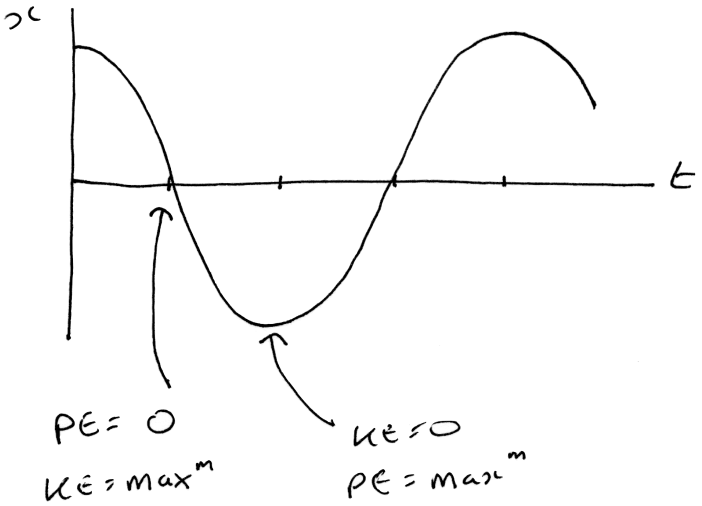
$$\langle KE\rangle = \frac{1}{2}m\langle \dot{x}^2\rangle = \frac{1}{2}mA^2\omega^2\langle \sin^2\omega t\rangle = \frac{1}{4}mA^2\omega^2 = \frac{1}{4}kA^2$$

$$\langle PE\rangle = \frac{1}{2}k\langle x^2\rangle = \frac{1}{2}kA^2\langle \cos^2\omega t\rangle = \frac{1}{4}kA^2$$

$$\langle E\rangle = \langle KE\rangle + \langle PE\rangle = \frac{1}{2}kA^2$$

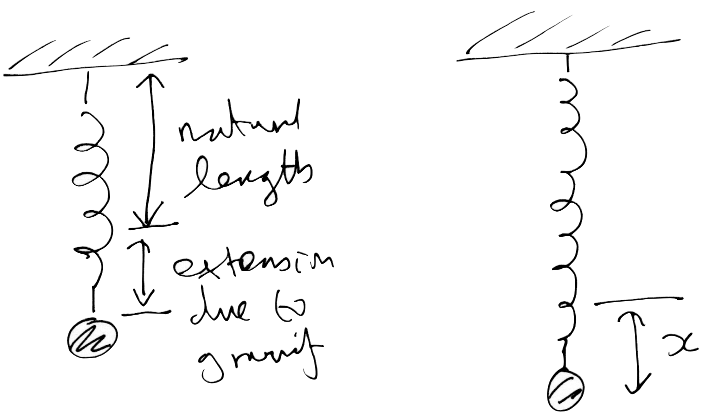
It is a standard result for the harmonic oscillator that the average kinetic and potential energies are the same. There is no surprise here when you consider that both are sinusoidal-squared functions, and that in the harmonic oscillator there is complete conversion between kinetic and potential energy.

Draw diagram showing where KE and PE have maxima and minima



Effect of gravity

Note that we are assuming that the displacements are small enough that the gravitational force is the same for all *x*. However, we have stretched the string a bit at the start by the balance of forces



King page 5 shows that this cancels out. The key thing is that the harmonic equation is linear in x, so that wherever you start from makes no difference.

Effect of the mass of the spring

If we also take account of the mass of the spring, the model is changed. Done for us in King as problem 1.12 (answers provided) and French pp 60ff.

If the mass of the object is *M*, and the mass of the spring is *m*, the angular frequency changes to

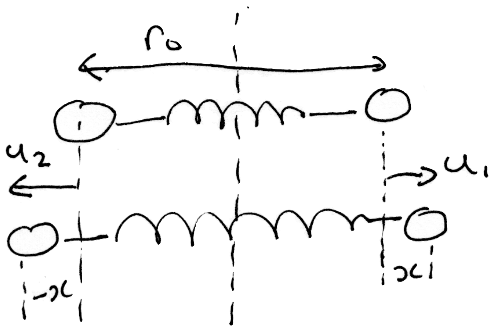
$$\omega^2 = \frac{k}{M + m/3}$$

The factor of 3 comes from integrating the kinetic energy of the whole spring, where the velocity of the spring depends on the square of the distance.

Vibrations of simple molecules

Case of two identical atoms in a molecule, eg H_2, N_2, O_2, F_2 etc

Picture of molecule showing the vibration



Energy

$$E = \frac{1}{2}k\left(u_1 - u_2\right)^2$$

Let's look at each atom

$$f_1 = -\frac{\partial E}{\partial u_1} = -k\left(u_1 - u_2\right) = m\ddot{u}_1$$
$$f_2 = -\frac{\partial E}{\partial u_2} = +k\left(u_1 - u_2\right) = m\ddot{u}_2$$

But we know the solutions. One is atoms move together, and one is they move exactly in opposite directions. The first doesn't involve

$$\begin{aligned}
f_1 &= -\frac{\partial E}{\partial u_1} = -k\left(u_1 - u_2\right) = m\ddot{u}_1 \\
f_2 &= -\frac{\partial E}{\partial u_2} = +k\left(u_1 - u_2\right) = m\ddot{u}_2 \\
u_1 &= a\cos\omega t \\
u_2 &= -a\cos\omega t \\
m\omega^2 a\cos\omega t &= k2a\cos\omega t \\
\omega^2 &= 2k/m
\end{aligned}$$

Note that the mass appears to be reduced by a factor of 2. This is the reduced mass, namely

$$\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{2}{m}$$

Some values for molecules

Molecule	*Mass (g/mol)*	*Frequency (THz)*	*Force constant (J/m²)*
H_2	1	132	571
D_2	2	93.5	573

Molecule	*Mass (g/mol)*	*Frequency (THz)*	*Force constant (J/m²)*
N_2	14	71	2313
O_2	16	47	1158
F_2	19	27	454
Cl_2	35.5	17	336
Br_2	79.9	9.5	236
I_2	126.9	6.2	160

Warning, units of frequency in literature can vary:

kHz = 10^3 Hz

MHz = 10^6 Hz

GHz = 10^9 Hz

THz = 10^{12} Hz

cm^{-1} from light spectroscopy, 100 cm^{-1} = 3 THz

eV often used in neutron spectroscopy, 1 eV = 0.242 GHz

meV, 1meV = 10^{-3} eV = 0.242 THz

Take the case of O_2:

$$k = (2\pi f)^2 \times m/2 = \left(2\pi \times 47\times 10^{12}\right)^2 \times \left(16\times 10^{-3}/6.022\times 10^{23}\right)/2 = 1158 \text{ J/m}^2$$

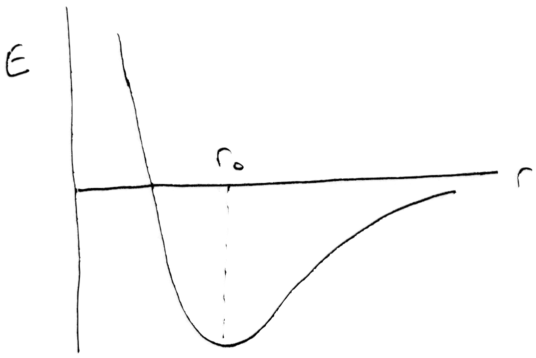
Note that frequencies go down with mass, expecting a root(mass) dependence. It is actually faster than this. For example, compare iodine with bromine. Square root of ratios of mass is

$$\sqrt{m_{I_2}/m_{Br_2}} = \sqrt{126.9/79.9} = 1.26$$
$$f_{I_2}/f_{Br_2} = 9.5/6.2 = 1.53$$

Ask class to now check that the results for H_2 and D_2 give the same force constant, should be about half of that for O_2

Model interatomic potential energy

Let's interpret this with a model interatomic potential:



Last year you looked at the Lennard-Jones potential:

$$E(r) = -4\varepsilon\left(\left(\frac{\sigma}{r}\right)^{6} - \left(\frac{\sigma}{r}\right)^{12}\right)$$

$$\frac{\mathrm{d}E}{\mathrm{d}r} = \frac{4\varepsilon}{\sigma}\left(6\left(\frac{\sigma}{r}\right)^{7} - 12\left(\frac{\sigma}{r}\right)^{13}\right)$$

$$\frac{\mathrm{d}E}{\mathrm{d}r} = 0 \Rightarrow 6\left(\frac{\sigma}{r}\right)^{7} - 12\left(\frac{\sigma}{r}\right)^{13} = 0 \Rightarrow \frac{\sigma}{r} = 2^{-1/6}$$

$$\frac{\mathrm{d}^2E}{\mathrm{d}r^2} = -\frac{4\varepsilon}{\sigma^2}\left(42\left(\frac{\sigma}{r}\right)^{8} - 156\left(\frac{\sigma}{r}\right)^{14}\right) = -\frac{4\varepsilon}{\sigma^2}\left(\frac{42}{2\times 2^{1/3}} - \frac{156}{4\times 2^{1/3}}\right) = \frac{72\varepsilon}{2^{1/3}\sigma^2}$$

So here you have two variables and three quantities, namely bond length, bond or dissociation energy, and the vibrational frequency. Does this work? You will see in exercise class.

Often for covalent bonds we use the Morse potential:

$$E(r) = E_0\left[\exp 2\alpha(r_0 - r) - 2\exp\alpha(r_0 - r)\right]$$

$$\frac{\mathrm{d}E}{\mathrm{d}r} = E_0\left[-2\alpha\exp 2\alpha(r_0 - r) + 2\alpha\exp\alpha(r_0 - r)\right]$$

$$= 0 \text{ when } r = r_0$$

$$E(r_0) = -E_0$$

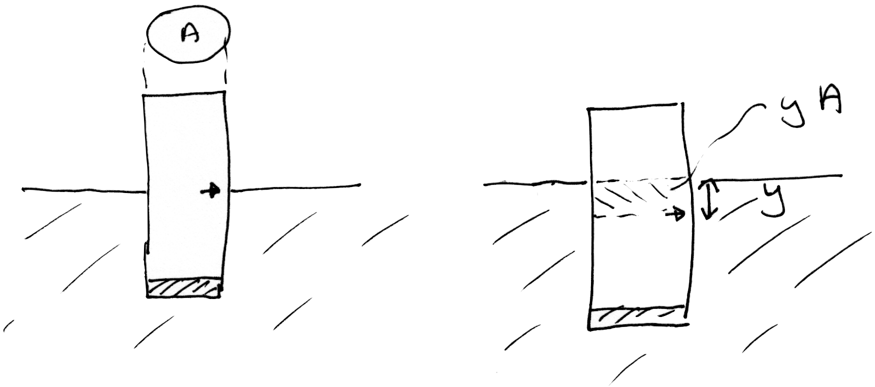
$$\frac{\mathrm{d}^2E}{\mathrm{d}r^2} = E_0\left[4\alpha^2\exp 2\alpha(r_0 - r) - 2\alpha^2\exp\alpha(r_0 - r)\right]$$

$$= 2\alpha^2 E_0 = k$$

So if we know the bond length and the molecular dissociation energy, we could evaluate the coefficient α from the known molecular vibrational frequencies. I will get you to look at these in next week's exercise class.

Floating object

Picture



Need to have uniform cross section area A, water density ρ, displacement y. Object has mass m. The mass of the water displaced by the motion is ρAy. For movement down, force is upwards due to displaced mass. So we have

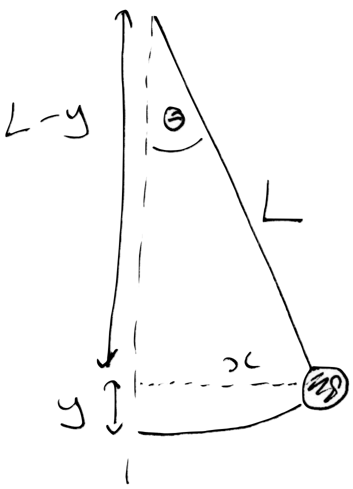
$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -g\rho A y$$
$$y = y_0 \cos \omega t$$
$$\omega^2 = g\rho A / m$$

Lets take some numbers. Suppose we have a light object of 0.1 kg, object of 1 cm radius, noting density of water is 1 g/cc, we have, with correct units

$$f = \frac{1}{2\pi}\sqrt{9.8 \times 10^3 \times \pi \times 10^{-4} / 0.1} = 0.88 \text{ Hz}$$

Simple Pendulum

Show diagram



Horizontal displacement causes vertical rise which creates an additional force

$$E = mgy$$

$$L - y = L\cos\theta \simeq L\left(1 - \frac{\theta^2}{2}\right)$$

$$y = \frac{1}{2}L\theta^2$$

$$x = L\sin\theta \simeq L\theta$$

$$y = \frac{1}{2L}x^2$$

$$E = \frac{mg}{2L}x^2$$

$$F = -\frac{\mathrm{d}E}{\mathrm{d}x} = -\frac{mg}{L}x = m\ddot{x}$$

$$m\ddot{x} + \frac{mg}{L}x = 0$$

$$x = A\cos\omega t$$

$$-m\omega^2 A\cos\omega t + \frac{mg}{L}A\cos\omega t = 0$$

$$-\omega^2 + \frac{g}{L} = 0$$

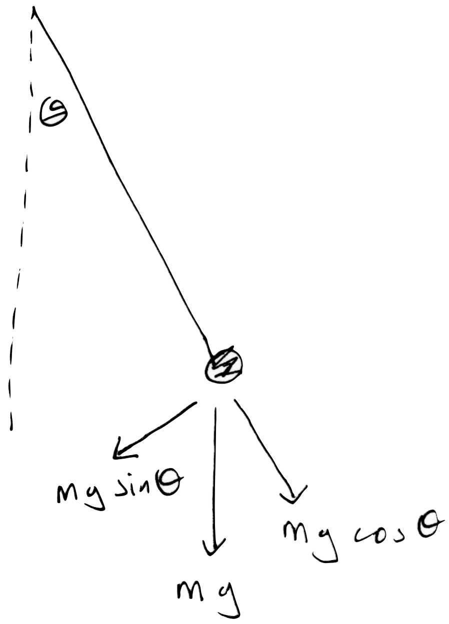
$$\omega = \sqrt{\frac{g}{L}}$$

Note that the frequency does not depend on mass. Nor on the initial conditions.

Comment on accuracy

1. We made small angle approximation, which is a standard trick in physics. It works when amplitudes are small, which is a common case.
2. In the case of the pendulum, the energy will increase at a slower rate that as the squared power for larger amplitude. Task to think why.

Rework in terms of angular coordinates, picture



Here we work in angular variables rather than the Cartesian coordinates from before. From the mechanics of rotations, where we define torque and moment of inertia, we have

$$I = mL^2$$

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mg\sin\theta L \approx -mg\theta L$$

$$L\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + g\theta = 0$$

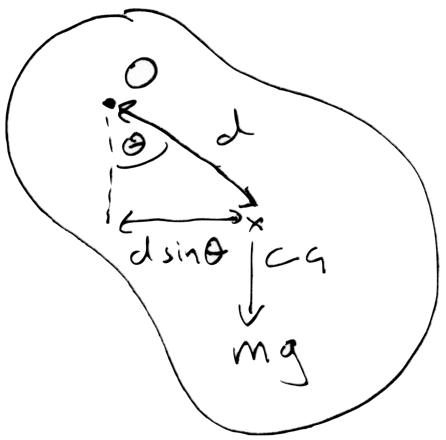
$$\theta = \theta_0 \cos(\omega t + \delta)$$

$$-L\omega^2 + g = 0$$

$$\omega = \sqrt{g/L}$$

which is exactly the same as before

Physical pendulum



Need to account for the mass distribution, so we work not in terms of the cartesian coordinates of the point mass, but in terms of the moment of inertia, angle of swing, and the torque.

As for the simple pendulum, we define a torque, which we call τ.

$$\tau = -mgd\sin\theta \approx mgd\theta = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{mgd}{I}\theta = 0$$

$$\omega^2 = \frac{mgd}{I}$$

This analysis is similar to what we saw for a simple pendulum, but now we have to take account of the various quantities. Note that if we put in the moment of inertia for the simple pendulum, things cancel and we get the same result as before.

Suppose we have a rod pivoting from one end, then we have

$$I = \frac{1}{3}mL^2$$

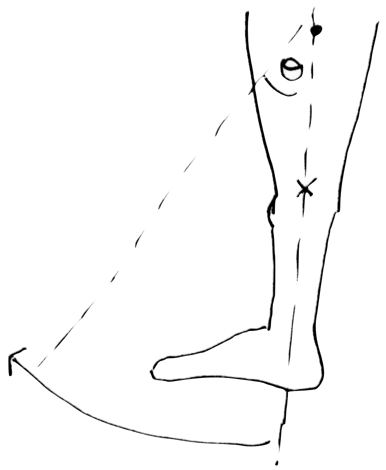
$$d = L/2$$

$$\omega^2 = \frac{mgd}{I} = \frac{mgL/2}{mL^2/3} = \frac{3g}{2L}$$

$$T = 2\pi\sqrt{2L/3g}$$

Swinging of leg and walking

Work in pairs here.



1. Swing a leg naturally and work out period
2. Estimate the length of your leg, called this *L*
3. Now treat the leg as a rigid rod, so that it's centre of mass is half way down, ie *L*/2, and its moment of inertia is that of a rigid rod = (1/3)mL^2. Compute the natural frequency in Hz, not radians per second, and hence the period.

So take someone whose leg is 0.8 m long. We have

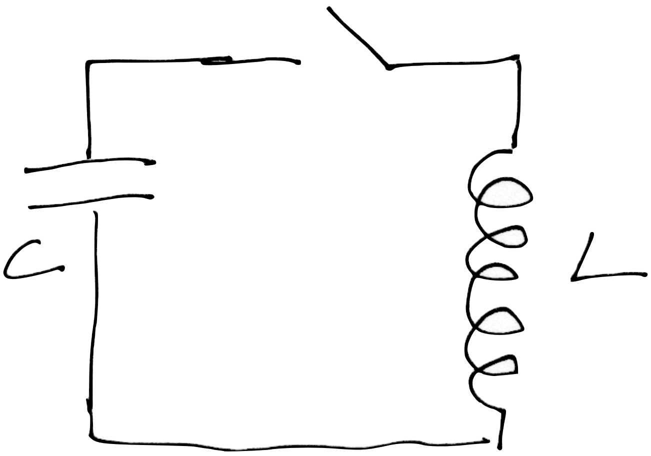
$$f = \frac{1}{2\pi}\sqrt{\frac{3g}{2L}} = \frac{1}{2\pi}\sqrt{\frac{3\times 9.8}{2\times 0.8}} = 0.68 \text{ Hz}$$
$$T = 1/f = 1/0.68 \approx 1.5 \text{ s}$$

Is this what you got?

If we have time, do an experiment

If you walk at 1 m/s, and if your stride is about 0.8 m, a stride will take 0.8 s. There are two strides in a period, so the period comes out as 1.6 s. Not bad eh! And note that it is independent of the mass of the person.

LC circuit



The idea is to charge the capacitor, then close the switch so that the capacitor can discharge through the indictor.

We use Kirchhoff's voltage law, which states that the sum of the voltages around the circuit is zero.

$$V_C = Q/C$$
$$V_L = L\frac{\mathrm{d}I}{\mathrm{d}t} = L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}$$
$$V_C + V_L = 0$$
$$Q/C + L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = 0$$
$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{1}{LC}Q = 0$$
$$Q(t) = Q_0\cos(\omega t + \delta)$$
$$\omega^2 = \frac{1}{LC}$$

Note that some sources will write these equations in terms of current.

Summary

1. Identified the differential equation that gives simple harmonic motion

2. For a number of physical systems, we have identified how the frequencies depend on the physical parameters. The results are not always intuitive
3. We have shown that the energy of a mechanical harmonic oscillator alternates between kinetic and potential. In the case of an LC circuit, the energy alternates between the energy in the capacitor and the energy in the inductor.