PHY217: Vibrations and Waves

Assessed Problem set 1

Issued: 5 November 2012

Deadline for submission: 5 pm Thursday 15th November, to the V&W pigeon hole in the Physics reception on the 1st floor of the GO Jones building.

Question 1

The central assumption behind the course so far is that the function describing the energy can be approximated as quadratic in the variable of interest. For two examples discuss the validity of this approximation. [10 marks]

We looked at a number of examples in the class, including the pendulum, the spring, electrical circuits, and a floating object. The case where we might expect the approximation of the energy being quadratic to be least-well obeyed is the pendulum, where the energy will scale as the sine of the rotation angle and we make the approximation that the angle is sufficiently small that we can make the approximation $\sin\theta \approx \theta$. This is actually not a good approximation when the amplitude is large, such as when swinging on a child's swing. For the spring however, because it is so well coiled, a significant expansion of the spring only leads to small changes in the material in the direction of the coil, so the quadratic approximation works well.

Question 2

1. Define the terms "under-damped", "over-damped" and "critically-damped". Give examples of when two of these properties can be exploited in real-world applications. [4 marks]

Damping arises from the existence of a force that depends on velocity, in that it doesn't operate on the equilibrium state (ie when the oscillating object is at rest), but only when the object moves. It has the effect of taking energy out of the system, and thus reduces the motion. In an under-damped system the damping is sufficiently light that it doesn't impede the oscillations, but merely reduces the amplitude over time. On the other hand, in an over-damped system, the damping is so strong that oscillations cannot be established, and instead the object moves slowly from its starting point to its equilibrium state. A critically-damped system is exactly between these two limits; it is at the point at which oscillations cannot be established, but has the fastest relaxation to equilibrium possible.

2. Explain the meaning of the quantities γ and ω_0 in the differential equation that describes a damped oscillator. [3 marks]

Formally γ and ω_0 are the prefactors of the time derivative of the coordinate and the coordinate respectively. They are obtained from the prefactors in the force expression divided by the mass.

3. Explain why consideration of forced oscillations requires the inclusion of damping in the model. [3 marks]

Because with forced oscillation, when the forcing oscillation has the resonant frequency it will give infinite amplitude. This is never going to be realised in reality because of effects such as damping, so we therefore have a need to take proper account of damping.

4. Describe the difference between steady-state motions of a forced oscillator and the transient state. Explain why the transient state is a valid part of the solution of the differential equation that describes forces and accelerations. [4 marks]

The steady-state solution is the solution that applies at long times, and is typically a vibration excited by the driving force. The solution to the unforced oscillator is also a valid contribution to the next solution. In the case of a damped oscillator, this solution decays with time, and hence is the solution at the start of the forced oscillation, and for this reason is called the transient solution. This solution will have a different frequency to that of the forcing oscillation, and there will be beating during the transient phase.

Question 3

Explain why the angular frequency of an oscillator is a property of the system when it is allowed to move freely when starting from an initial state, but is imposed when there is a continuous sinusoidal applied force. In which case is the amplitude of motion established by the initial conditions? [10 marks]

The equation of motion for an unforced system will have a solution that gives a single angular frequency (for a single oscillator; there will be as many frequencies as normal modes in the system, and as many normal modes as oscillating objects). The angular frequency reflects the restoring force of the system. The actual amplitude is set by the initial conditions. When you apply an oscillating force, whilst the system would like to oscillate at its natural frequency, it is required to oscillate at the frequency of the applied force, and the steady state amplitude is set entirely by the applied force. Think of what happens at very low forcing frequency; at this limit it is obvious that the object moves exactly with the forcing oscillation.

Question 4

1. Define the figure of merit Q for a damped oscillator (unforced or forced). [5 marks]

Formally it is given as $Q = \omega_0 / \gamma$. Low damping leads to a high value of Q.

2. Explain why Q is a useful quantity in the study of forced oscillations. [5 marks]

Q does a number of things; it quantifies the sharpness of the resonance of the forced oscillation (both amplitude and power absorption), and it quantifies the rate of energy loss of the damped oscillation.

Question 5

Consider a hydrogen fluoride molecule (atomic mass of H is 1 g/mole, and of F is 19 g/mole).

1. Write the energy of the system in terms of the displacements of both atoms. [3 marks]

$$E = \frac{1}{2}k\left(u_1 - u_2\right)^2$$

2. Write the equation for force = mass × acceleration as a differential equation for both atoms, obtaining the force from the differential of the energy. [3 marks]

$$F_\mathrm{H} = -\frac{\mathrm{d}E}{\mathrm{d}x_\mathrm{H}} = -k\left(x_\mathrm{H} - x_\mathrm{F}\right) = m_\mathrm{H}\dot{u}_\mathrm{H}$$

$$F_\mathrm{F} = -\frac{\mathrm{d}E}{\mathrm{d}x_\mathrm{F}} = +k\left(x_\mathrm{H} - x_\mathrm{F}\right) = m_\mathrm{F}\dot{u}_2$$

3. Show that the following equations for the displacements of the atoms are solutions of the differential equation

$$x_{\mathrm{H}} = \frac{x_0}{m_{\mathrm{H}}}\cos\omega t$$

$$x_{\mathrm{F}} = -\frac{x_0}{m_{\mathrm{F}}}\cos\omega t$$

and hence obtain an equation for the angular frequency ω. [4 marks]

$$\ddot{x}_{\mathrm{H}} + \frac{k}{m_{\mathrm{H}}}\left(x_{\mathrm{H}} - x_{\mathrm{F}}\right) = 0 \Rightarrow -\omega^2 \frac{x_0}{m_{\mathrm{H}}}\cos\omega t + \frac{k}{m_{\mathrm{H}}}\left(\frac{x_0}{m_{\mathrm{H}}} + \frac{x_0}{m_{\mathrm{F}}}\right)\cos\omega t = 0$$

$$\ddot{x}_{\mathrm{F}} - \frac{k}{m_{\mathrm{F}}}\left(x_{\mathrm{H}} - x_{\mathrm{F}}\right) = 0 \Rightarrow -\omega^2 \frac{x_0}{m_{\mathrm{F}}}\cos\omega t - \frac{k}{m_{\mathrm{F}}}\left(\frac{x_0}{m_{\mathrm{H}}} + \frac{x_0}{m_{\mathrm{F}}}\right)\cos\omega t = 0$$

$$\Rightarrow \omega^2 = k\left(\frac{1}{m_{\mathrm{H}}} + \frac{1}{m_{\mathrm{F}}}\right)$$

4. The vibration frequency is measured as 124 THz; use this result to obtain the effective spring constant of the bond between the H and F atoms. [3 marks]

$$\omega^2 = \left(2\pi \times 124 \times 10^{12}\right)^2 = k\left(\frac{1}{1\times10^{-3}} + \frac{1}{19\times10^{-3}}\right)\times 6.0222\times10^{236}$$

$$k = \frac{6.07\times10^{29}}{6.34\times10^{26}} = 957 \text{ J/bond/m}^2$$

5. Predict the frequency of the molecule if the hydrogen atom is replaced by a deuterium atom (atom mass 2 g/mole). [3 marks]

$$f = \frac{1}{2\pi}\sqrt{957\left(\frac{1}{2\times10^{-3}} + \frac{1}{19\times10^{-3}}\right)\times 6.0222\times10^{23}} = 89.8 \text{ THz}$$

Question 6

Find solutions of the following differential equations:

1. $\ddot{y} + 16y = 0$ [3 marks]

   $y = y_0 \cos 4t$

2. $\ddot{y} + 4\dot{y} + 16y = 0$ [3 marks]

   $y = A\exp(-2t)\cos\left(\sqrt{12}t\right)$

3. $\ddot{y} + 8\dot{y} + 16y = 0$ [3 marks]

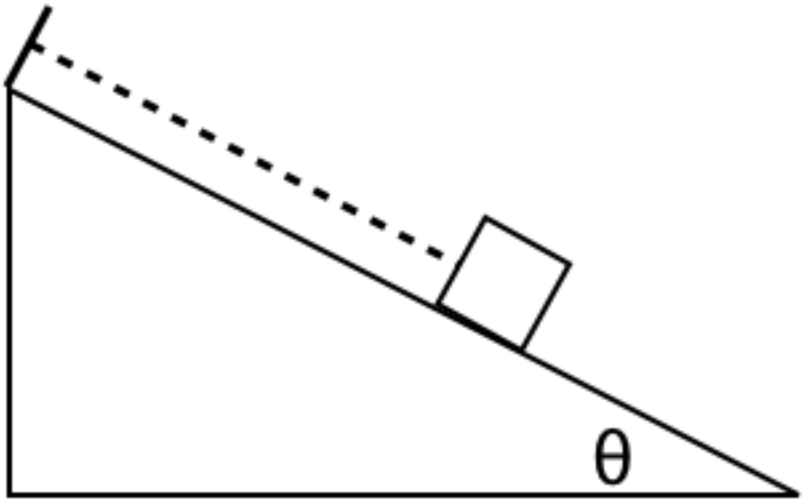
   $y = \exp(-4t)\left(A + Bt\right)$

4. $\ddot{y} + 16\dot{y} + 16y = 0$ [3 marks]

   $y = \exp(-4t)\left(A\exp\left(\sqrt{48}t\right) + B\exp\left(-\sqrt{48}t\right)\right)$

Question 7

A mass m is attached to a massless spring with a spring constant k. The mass can slide along a frictionless plane inclined as an angle θ to the ground (as illustrated below).



1. Write an equation for the energy as the mass is displaced from its equilibrium position, assuming that the only force acting is gravitational. [4 marks]

Call the displacement along the incline x. The energy is

$$E = -mgx\sin\theta + \frac{1}{2}kx^2$$

2. Derive the differential equation describing the force and acceleration. [4 marks]

We need to calculate the force but we need to ensure we oscillate about equilibrium, ie where the force is zero.

$$F = -\frac{\mathrm{d}E}{\mathrm{d}x} = mg\sin\theta - kx = 0$$
$$x = mg\sin\theta / k = x_0$$

We need to create our equations considering oscillations about this point, so we can define a new variable u defining motion from the equilibrium point. The equations become

$$F = -\frac{\mathrm{d}E}{\mathrm{d}x} = mg\sin\theta - k(x_0 + u) = 0$$
$$kx_0 = mg\sin\theta$$
$$\Rightarrow F = -ku = m\ddot{u}$$

3. Show the the period of motion is independent of the angle θ. [4 marks]

Done above – the final equation for force doesn't contain the angle.

Question 8

An object of mass m = 0.2 kg is hung from a spring whose spring constant is 80 $\mathrm{Nm^{-1}}$. The body is subject to a resistive force given by $-bv$, where v is its velocity and b = 4 $\mathrm{Nm^{-1}s}$.

1. Set up the differential equation of motion for free oscillations of the system, and find the period of such oscillations. [4 marks]

$$\ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=0$$

The angular frequency in the case of light damping is

$$\omega^2=\omega_0^2-\gamma^2/4$$

$$=\frac{k}{m}-\left(\frac{b}{m}\right)^2/4=\frac{80}{0.2}-\left(\frac{4}{0.2}\right)^2/4=400-100=300$$

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{\sqrt{300}}=0.36\text{ s}$$

2. The object is subjected to a sinusoidal force given by $F(t) = F_0 \sin(\omega t)$, where $F_0 = 2$ N and $\omega = 30$ rad/s. In steady state, what is the amplitude of the forced oscillation? [4 marks]

The equation here is

$$A(\omega)=\frac{F_0/m}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}=\frac{2/0.2}{\sqrt{\left(400-900\right)^2+(4/0.2)^2 30^2}}=1.28\text{ cm}$$

3. Instead of a driving force, we now oscillate the end of the spring at the top end vertically with a harmonic displacement $X = X_0 \sin(\omega t)$. Set up the differential equation of motion for this driven oscillator. [4 marks]

The equation here is

$$\ddot{x}+\gamma\dot{x}+\omega_0^2 x=\frac{k}{m}\eta_0\sin\omega t$$

4. What is the amplitude of the mass in steady state for $\omega = 0$, 30 and 300 rad/s, if $X_0 = 0.5$ cm in each case? [4 marks]

The equation here is

$$A(\omega)=\frac{\eta_0 k/m}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}=\frac{0.5\times 80/0.2=200}{\sqrt{\left(400-\omega^2\right)^2+400\omega^2}}$$