Vibrations and waves: exercise sheet 8

Long chain of coupled pendulums

Picture an infinite number of pendulums, where each pendulum is connected to its two neighbours by a spring with force constant *J*. Each pendulum has length *L* and supports a mass *m*. Neighbouring pendulums are separated by distance *a*. Consider the case where the masses move in the direction of the chain.

Write the energy for this system in terms of both the spring and gravitational energies. Compute the force acting on any selected mass when it and it's neighbours move.

$$E_j = \frac{1}{2}J\left(u_j - u_{j-1}\right)^2 + \frac{1}{2}J\left(u_j - u_{j+1}\right)^2 + \frac{mg}{2L}u_j^2$$

$$F_j = -\frac{\partial E}{\partial u_j} = -J\left(u_j - u_{j-1}\right) - J\left(u_j - u_{j+1}\right) - \frac{mg}{L}u_j$$

Assume the travelling wave solution

$$u_j = \tilde{u}\exp\left(i(kx - \omega t)\right)$$

Show that the dispersion curve for this system is described by the equation

$$\omega^2 = \frac{4J}{m}\sin^2(ka/2) + \frac{g}{L}$$

$$-J\left(u_j - u_{j-1}\right) - J\left(u_j - u_{j+1}\right) - \frac{mg}{L}u_j = -J\tilde{u}\exp\left(i(kja - \omega t)\right)\left(2 - \exp(-ika) + \exp(+ika)\right) - \frac{mg}{L}\tilde{u}\exp\left(i(kja - \omega t)\right)$$

$$= m\ddot{u}_j$$

$$= -m\omega^2\tilde{u}\exp\left(i(kja - \omega t)\right)$$

$$\Rightarrow J\left(2 - \exp(-ika) + \exp(+ika)\right) + \frac{mg}{L} = m\omega^2$$

$$\Rightarrow \omega^2 = \frac{4J}{m}\sin^2(ka/2) + \frac{g}{L}$$

Sketch the function for ω. Highlight points on the dispersion curve that correspond to standing waves.

These are the points at $k = 0$ and $k = \pi/a$, where the dispersion curve has zero slope (at a maximum or minimum)

Rationalise this result in the limits of zero value of *k*, and zero value of *J*.

In both limits, the first term in the function is equal to zero, and we just have the solution for a single pendulum. The point of $k = 0$ is that at the point all pendulums should be doing the same thing. The point of $J = 0$ is that regardless of the wave vector, the oscillators are not coupled and are therefore acting as independent oscillators.

Why do we not find the equivalent of a sound wave?

A wave will travel through this system, but it won't be a sound wave because it doesn't have a frequency that falls to zero at infinite wavelength. The limit of zero value of k gives a non-zero frequency value. Physically, a zero frequency corresponds to a very low force required to perform a uniform shift of many oscillators together because the springs between neighbours are not stretched or compressed. In our case, to move a single oscillator now costs energy, so shifting a block of oscillators will cost energy regardless of whether springs are stretched or compressed.