Vibrations and waves: exercise sheet 6

Double pendulum

Consider a pendulum consisting of a string and two weights, one at distance L_1 from the top and the other at distance L_2 below the upper weight. Both weights have mass m.

1. Write the energy of the system when both weights swing, in terms of the horizontal displacements.

The upper pendulum swings with an angle t. Translating into x and y coordinates gives

$$y_1 = \frac{1}{2}L_1\theta_1^2 \quad ; \quad x_1 = L_1 \sin\theta_1 \simeq L_1\theta_1 \quad ; \quad y_1 = \frac{1}{2L_1}x_1^2$$

$$y_2 = y_1 + \frac{1}{2}L_2\theta_2^2 \quad ; \quad x_2 - x_1 = L_2 \sin\theta_2 \simeq L_2\theta_2 \quad ; \quad y_2 = \frac{1}{2L_1}x_1^2 + \frac{1}{2L_2}\left(x_2 - x_1\right)^2$$

$$E = mg\left(y_1 + y_2\right) = mg\left(\frac{1}{L_1}x_1^2 + \frac{1}{2L_2}\left(x_2 - x_1\right)^2\right)$$

2. Write the equations of motion of both weights from the force and accelerations

$$F_1 = -\frac{\mathrm{d}E}{\mathrm{d}x_1} = -\frac{2mg}{L_1}x_1 + \frac{mg}{L_2}\left(x_2 - x_1\right) = m\ddot{x}_1$$

$$F_2 = -\frac{\mathrm{d}E}{\mathrm{d}x_2} = -\frac{mg}{L_2}\left(x_2 - x_1\right) = m\ddot{x}_2$$

3. Take the two solutions $x_1 = a_1 \cos\omega t$ and $x_2 = a_2 \cos\omega t$. Substitute into the general solution and write in matrix form

$$-\omega^2 a_1 + \frac{2g}{L_1}a_1 - \frac{g}{L_2}\left(a_2 - a_1\right) = 0$$

$$-\omega^2 a_2 + \frac{g}{L_2}\left(a_2 - a_1\right) = 0$$

$$\begin{pmatrix} \frac{2g}{L_1} + \frac{g}{L_2} - \omega^2 & -\frac{g}{L_2} \\ -\frac{g}{L_2} & \frac{g}{L_2} - \omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

4. Calculate expressions for the two normal mode frequencies, and then simplify for the case that $L_2 = L_1/2$.

$$\begin{vmatrix} \dfrac{2g}{L_1}+\dfrac{g}{L_2}-\omega^2 & -\dfrac{g}{L_2} \\ -\dfrac{g}{L_2} & \dfrac{g}{L_2}-\omega^2 \end{vmatrix}=0$$

$$\left(\frac{2g}{L_1}+\frac{g}{L_2}-\omega^2\right)\left(\frac{g}{L_2}-\omega^2\right)-\left(\frac{g}{L_2}\right)^2=0$$

$$\omega^4-\omega^2 2g\left(\frac{1}{L_1}+\frac{1}{L_2}\right)+\frac{g}{L_2}\left(\frac{2g}{L_1}+\frac{g}{L_2}\right)-\left(\frac{g}{L_2}\right)^2=\omega^4-\omega^2 2g\left(\frac{1}{L_1}+\frac{1}{L_2}\right)+\frac{2g^2}{L_1L_2}=0$$

$$\omega^2=g\left(\frac{1}{L_1}+\frac{1}{L_2}\right)\pm\sqrt{g^2\left(\frac{1}{L_1}+\frac{1}{L_2}\right)^2-\frac{2g^2}{L_1L_2}}$$

$$=g\left(\frac{1}{L_1}+\frac{1}{L_2}\right)\pm g\sqrt{\left(\frac{1}{L_1}\right)^2+\left(\frac{1}{L_2}\right)^2}$$

$$\omega^2=g\left(\frac{1}{L_1}+\frac{2}{L_1}\right)\pm g\sqrt{\left(\frac{1}{L_1}\right)^2+\left(\frac{2}{L_2}\right)^2}=\frac{g}{L_1}\left(3\pm\sqrt{5}\right)$$