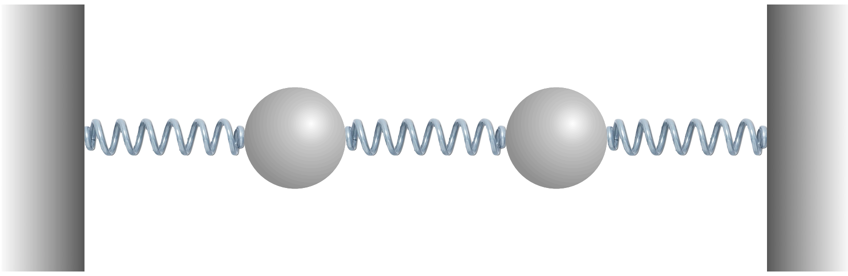
Vibrations and waves: exercise sheet 5

Coupled oscillators

Consider the following simple example



Each ball has mass m, and each spring has force constant k. Assign a displacement x_1 and x_2 to each of the two balls along the direction normal to the two end walls. The two ends are fixed.

1. Write an equation for the potential energy of this system for any displacement of each ball.

$$E = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}kx_2^2$$

2. Differentiate the potential energy to derive an equation for the instantaneous force on each atom for any displacement of each ball.

$$F_1 = -\frac{\mathrm{d}E}{\mathrm{d}x_1} = -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2$$
$$F_2 = -\frac{\mathrm{d}E}{\mathrm{d}x_2} = +k(x_1 - x_2) - kx_2 = -2kx_2 + kx_1$$

3. By equating the force to a mass x acceleration, write a differential equation of motion for each ball.

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

4. Assume a general solution of the form

$$x_1 = C_1 \cos \omega t$$
$$x_2 = C_2 \cos \omega t$$

   and substitute these solutions into the differential equations (and replace k/m by the square of the spring angular frequency).

$$-C_1\omega^2 + 2\frac{k}{m}C_1 - \frac{k}{m}C_2 = 0$$

$$= -C_1\omega^2 + 2\omega_0^2 C_1 - \omega_0^2 C_2$$

$$-C_2\omega^2 + 2\frac{k}{m}C_2 - \frac{k}{m}C_1 = 0$$

$$= -C_2\omega^2 + 2\omega_0^2 C_2 - \omega_0^2 C_1$$

5. Recast these two equations as a matrix equation.

$$\begin{pmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

6. By finding the eigenvalues of the square matrix, obtain equations for the angular frequencies of the two normal modes.

$$\left(2\omega_0^2 - \omega^2\right)^2 - \left(\omega_0^2\right)^2 = 0$$

$$\Rightarrow 2\omega_0^2 - \omega^2 = \pm\omega_0^2$$

$$\omega^2 = \begin{cases} \omega_0^2 \\ 3\omega_0^2 \end{cases}$$

7. Use these two eigenvalues to calculate the relative displacements of the two balls for each normal mode.

$$\begin{pmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} \omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \omega_0^2\left(C_1 - C_2\right) \\ -\omega_0^2\left(C_1 - C_2\right) \end{pmatrix} = 0$$

$$\Rightarrow C_1 = C_2$$

$$\begin{pmatrix} -\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega_0^2 \end{pmatrix} \times \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} -\omega_0^2\left(C_1 + C_2\right) \\ -\omega_0^2\left(C_1 + C_2\right) \end{pmatrix} = 0$$

$$\Rightarrow C_1 = -C_2$$

So the two normal modes have the balls moving in phase (lower frequency; no stretching of middle spring) and out of phase (higher frequency; stretching of middle spring). No surprises here!