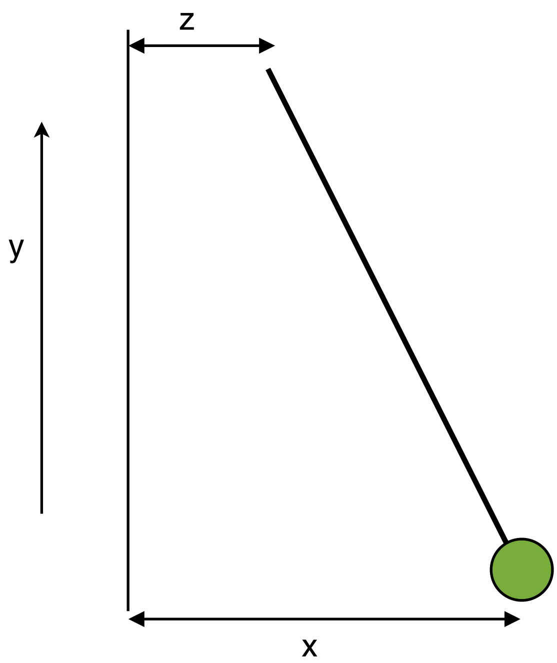
Vibrations and waves: exercise sheet 3

Forced pendulum

Consider the following image of a pendulum (mass *m*) subject to the fixed end being moved from side to side.



Assume that *z* represents my hand moving back and forth.

1. Use the formalism we established in week 2 to give an expression for the energy in terms of x – z (small angle approximation), and hence the equation for a simple harmonic oscillation in terms of the acceleration of *x*.

$$E = mgy$$

$$L - y = L\cos\theta \simeq L\left(1 - \frac{\theta^2}{2}\right)$$

$$y = \frac{1}{2}L\theta^2$$

$$x = L\sin\theta \simeq L\theta$$

$$y = \frac{1}{2L}(x-z)^2$$

$$E = \frac{mg}{2L}(x-z)^2$$

$$F = -\frac{\mathrm{d}E}{\mathrm{d}x} = -\frac{mg}{L}(x-z) = m\ddot{x}$$

$$m\ddot{x} + \frac{mg}{L}(x-z) = 0$$

2. Add a frictional term and scale by mass.

$$m\ddot{x}+b\dot{x}+\frac{mg}{L}(x-z)=0$$

$$\ddot{x}+\gamma\dot{x}+\frac{g}{L}(x-z)=0$$

$$\ddot{x}+\gamma\dot{x}+\omega_0^2(x-z)=0$$

3. Now write z as a sinusoidal (cosine) tern with amplitude z_0. Compare this equation with the equation for the forced harmonic oscillator we derived in week 3. What is the equivalent of the previous 'variable' $a = F/k$? Does it have the same units?

$$\ddot{x}+\gamma\dot{x}+\omega_0^2 x=\omega_0^2 z_0\cos\omega t$$

z_0 is the equivalent of F/k. It has units of length, which is the same as F/k.

4. From the algebraic solution to the equation of motion (do not derive but write down by analogy with the equations developed in week 3), state how does the amplitude z_0 controls the amplitude of the motion of the pendulum when my hand moves very slow?

$$x=A(\omega)\exp\left(i(\omega t-\delta)\right)$$

$$A(\omega)=\frac{\omega_0^2 z_0}{\left(\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2\right)^{1/2}}$$

$$A(\omega\to 0)=z_0$$

5. Similarly, How is the amplitude modified when my hand moves at the natural frequency of the pendulum?

$$A(\omega)=\frac{\omega_0^2 z_0}{\left(\left(\omega_0^2-\omega_0^2\right)^2+\gamma^2\omega_0^2\right)^{1/2}}$$

$$A(\omega\to 0)=z_0\omega_0/\gamma=z_0 Q$$

6. How does the phase of the motion of the pendulum change with the phase of the motion of my hand as my hand increases its speed of motion?

At low frequency, the phase angle is zero (the pendulum moves with my hand). At the natural frequency the phase angle is π/2. This increases towards π for very fast motion.

7. Show that the pendulum will hardly move at all in the limit of my hand moving extremely fast.

$$A(\omega\to\infty)=\frac{\omega_0^2 z_0}{\left(\omega^4+\gamma^2\omega^2\right)^{1/2}}\to 0$$