BSc/MSci Midterm Test (Solutions)

PHY-217 Vibrations and Waves

Time Allowed: 40 minutes

Date: 1 November 2007

Time: 15:10

Answer ALL questions in section. There are 40 marks in total.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

NUMERIC CALCULATORS ARE PERMITTED IN THIS EXAMINATION.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

1 A spring with spring constant k is suspended vertically. A mass m = 1.2 kg is attached to its free end and at equilibrium, the spring extends by Δ = 0.05 m. The mass is then pulled down a further 0.15 m, released from rest and found to execute simple harmonic motion (g = 9.81 m s^{-2}).

(a) Obtain the equation of motion, and hence show that the displacement $x(t)$ from its resting position has the general form

$$x(t) = A_0 \cos(\omega_0 t + \varphi)$$ [10 *marks*]

For such a system, the restoring force is proportional to how much the spring is stretched.
From its unstretched position the spring is initially stretched to its equilibrium length by some amount Δ = 0.05 m.
Thus,
$F_{eq} = -k\Delta$ (**As the up force from the spring is balanced by gravity $k\Delta = mg$**)

The spring is now stretched by an additional amount x = 0.15 m
The total force in the spring is the sum of these two forces
$F_{\text{spring}} = -k(\Delta + x)$
Hence, $-k\Delta - kx = mg + m\ddot{x}$
But, $k\Delta = mg$
Therefore, $-mg = kx = mg + m\ddot{x}$
Therefore, $m\ddot{x} = -kx$ (note to get full marks you must show this, not just state it)
Finally divide by m to get the SHO-type equation
$\ddot{x} = -\omega_0^2 m,$ where $\omega_0^2 = k/m$
This is our equation of motion.

We are told $x(t) = A_0 \cos(\omega_0 t + \varphi)$ **is a general solution.**

Our equation of motion is a 2nd order differential equation in terms of x. Therefore, to test the suggested solution we differentiate twice and compare the result with the equation of motion.

$$x(t) = A\cos(\omega t + \varphi)$$
$$\dot{x}(t) = -\omega \sin(\omega t + \varphi)$$
$$\ddot{x}(t) = -\omega^2 A\cos(\omega t + \varphi)$$
$$but, \quad A\cos(\omega t + \varphi) = x$$
$$\therefore \ddot{x} = -\omega^2 x$$
As required.

Note that I explained what I was doing and why I was doing it at every stage in this exercise. Note also that the first part of the question asked you to OBTAIN the equation of motion. This implies that marks are gained for demonstrating you know how to obtain it, not just for the final answer. Similarly, the 2nd part asks you to SHOW that the given equation is a solution. Similarly there are marks for how you show. Correct answers without explanations cannot gain full marks in this course as there are marks for the method not just the answer.

(b) Calculate the numerical values of the following:

(i) the amplitude and the constant phase angle; [5 *marks*]

We are told that $x(t) = A_0 \cos(\omega_0 t + \varphi)$ **.**
We are also told that initially, $x_0 = x(t = 0) = 0.15$ **m.**

Therefore, $x(t = 0) = 0.15 = A_0 \cos\varphi$

We also know the mass was released from rest (*i.e.*, its initial velocity = 0)

Therefore, $\dot{x}(t = 0) = 0 = -\omega_0 A_0 \sin\varphi$ **.**

Since ω_0 **and** A_0 **must both be non-zero, sin** φ **must = 0 , which implies** $\varphi = 0$**.**

Therefore, since $\varphi = 0$**,**

$$x(t = 0) = 0.15 = A_0 \cos(0)$$
$$\therefore A_0 = 0.15\,\mathrm{m}$$

Thus, $A_0 = 0.15$ **m and** $\varphi = 0$ **radians**

(ii) the frequency and the period; [5 *marks*]

At equilibrium. the extension of the spring by the mass is balanced by gravity (*i.e.*, $k\Delta = mg$**)**

Therefore,

$$\frac{k}{m} = \frac{g}{\Delta}$$

But

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{9.81}{0.05}} = 14.0\,\text{radians}\,s^{-1}.$$

Now, $\nu = 2\pi/\omega_0 = 2.23$ **s**$^{-1}$**.**

And $T = 1/\nu = 1/2.23 = 0.45$ **s.**

(iii) the maximum and minimum speed of the mass; [4 *marks*]

First obtain an expression for the general speed.

$x(t) = A_0 \cos(\omega_0 t + \varphi)$

Therefore, the velocity $\dot{x}(t) = -\omega_0 A_0 \sin(\omega_0 t + \varphi)$

But the speed $= |\dot{x}(t)| = \omega_0 A_0 \sin(\omega_0 t + \varphi)$

We need the maximum speed, and since for our given system A_0 and ω_0 are fixed, we need the maximum of $\sin(\omega_0 t + \varphi)$. Now $\sin(\omega_0 t + \varphi)$ varies between 1 and -1. Thus its maximum is 1.
Therefore, the maximum speed is $\omega_0 A_0 = 14.0 \times 0.15 = 2.1\ \mathrm{ms}^{-1}$.

Conversely, the minimum speed occurs when $\sin(\omega_0 t + \varphi)$ is half way between 1 and -1; *i.e.*, when $\sin(\omega_0 t + \varphi) = 0$, which occurs when $(\omega_0 t + \varphi) = 0$.
Therefore, the minimum speed is $\omega_0 A_0 \sin 0 = 0\ \mathrm{ms}^{-1}$

(iv) the maximum kinetic energy, and hence, giving an argument based on the conservation of energy, the total energy E. [4 *marks*]

$\mathrm{KE}_{max} = \frac{1}{2} m v_{max}^2$. And $v_{max} = 2.1\ \mathrm{ms}^{-1}$.

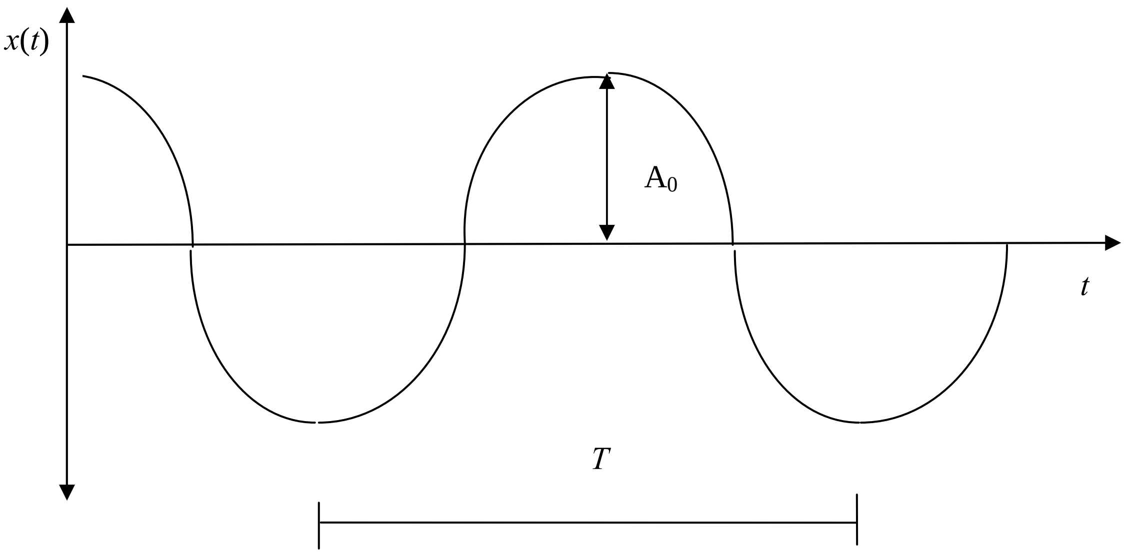
Therefore, $\mathrm{KE}_{max} = \frac{1}{2} \times 1.2 \times 2.1^2 = 2.7$ J

Note the answer is correctly 2.7 J; not 2.65 J, or anything else more 'accurate.' This is consistent with the least accurate parameter, which has only two significant figures. In the final exam you will loose marks for quoting answers too accurately).

The maximum KE occurs when the velocity is at a maximum, and the minimum KE occurs when the velocity is a minimum. Since the minimum velocity is 0 ms^{-1}, the minimum KE = 0 J.

By the conservation of energy principle, the total energy is the sum of KE and PE, and this must remain constant throughout the cycle. Since the KE falls to a minimum of 0 J, the minimum of PE must also be 0 J. Therefore, at KE_{max}, PE = 0 J. Thus, the total energy must be equal to KE_{max} = 2.7 J.

2 For the oscillator discussed in Question 1, sketch $x(t)$. Indicate on your sketch the amplitude and the period.



(1 mark for T, 1 mark for A_0, and one mark for a cosine wave with max at $t = 0$)

3 The displacement $x(t)$ from Question 1 is a solution of the simple harmonic oscillator equation and can be obtained from the complex variable $z(t)$ satisfying the same equation of motion.

(a) Write down this equation and also the mathematical form of its general complex solution $z(t)$. [2 *mark*]

Complex form of the equation of motion: $\ddot{Z} = -\omega_0^2 z$

Complex form of its solution: $z(t) = A_0 e^{i(\omega_0 t + \varphi)}$

(b) The variable $z(t)$ can be depicted in an Argand (or phasor) diagram as a rotating vector. Draw an Argand diagram showing this vector at arbitrary time t, indicating its magnitude, the phase angle of the vector, and the sense and magnitude of its angular velocity. Also indicate the physical displacement $x(t)$. [7 *marks*]