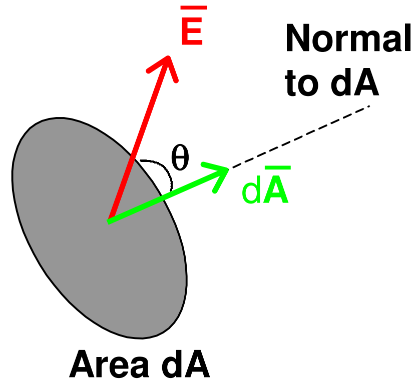
ELECTRIC FLUX AND GAUSS'S LAW
(*Young & Freedman* Chap. 23)
(*Ohanian* Chap. 24)

Electric flux, Φ

Consider a small flat area dA.

Let $\bar{\mathbf{E}}$ be the electric field at its centre.

Assume that dA is so small that $\bar{\mathbf{E}}$ can be regarded as uniform over the whole of dA.



Definition: The ELECTRIC FLUX, $d\Phi$ through the area dA is the product of dA and the normal component of $\bar{\mathbf{E}}$.

or $d\Phi = (E\cos\theta)dA$ so $d\Phi = \bar{\mathbf{E}} \cdot d\bar{\mathbf{A}}$

where $d\bar{\mathbf{A}}$ is the NORMAL VECTOR of the area dA:

Magnitude of $d\bar{\mathbf{A}}$ = dA
Direction of $d\bar{\mathbf{A}}$ is perpendicular to dA

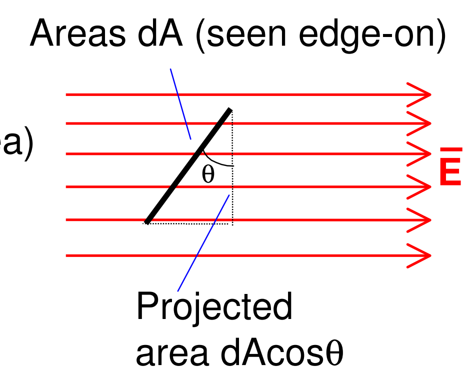
Note: 1. Electric flux is a SCALAR.

2. The electric flux through area dA can be thought of as the number of field lines crossing dA.

$d\Phi = E(dA\cos\theta)$

= (No. of lines/unit area)(Projected area)

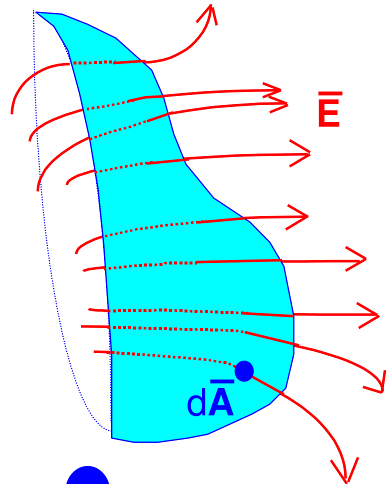
= No. of lines crossing dA



So far we've considered a UNIFORM field passing through a FLAT surface.

General case

Consider a NON-UNIFORM field $\overline{\mathbf{E}}$
passing through a NON-FLAT surface, A.

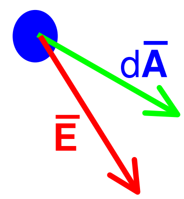


Divide A into many small elements (patches) such as dA.

dA is small $\Rightarrow$ (i) it's approximately flat;

(ii) $\overline{\mathbf{E}}$ is uniform over dA.

$\Rightarrow$ Flux through dA is $d\Phi = \overline{\mathbf{E}} \cdot d\overline{\mathbf{A}}$



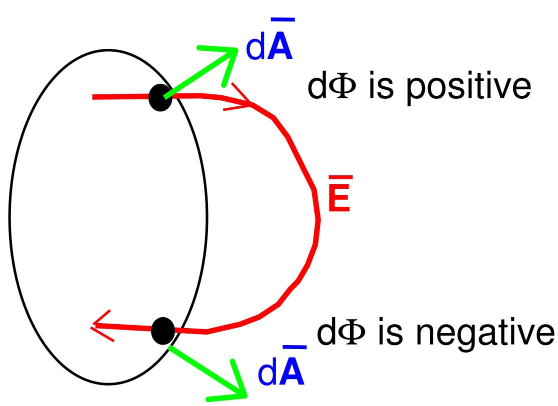
To find the total flux through the whole surface, we INTEGRATE over the whole of the area A:

$$\Phi = \int_A \overline{\mathbf{E}} . d\overline{\mathbf{A}}$$

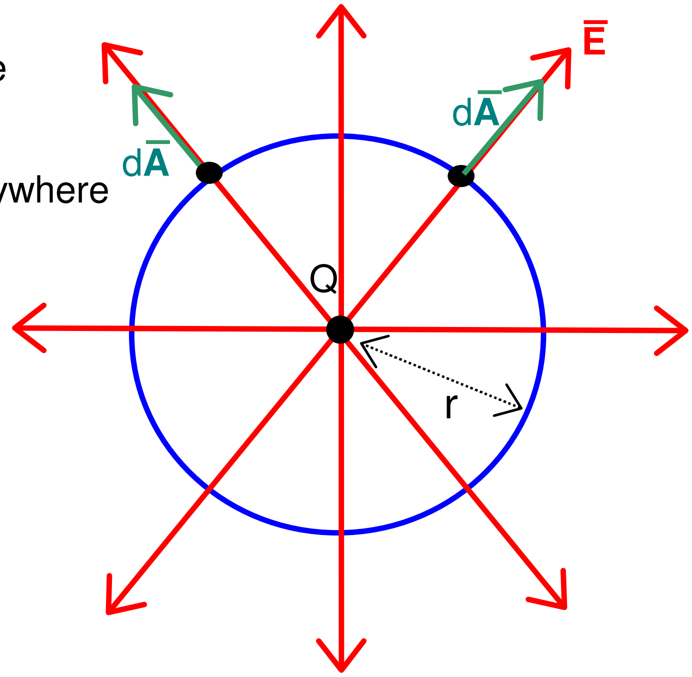
Note: 1. By convention $d\overline{\mathbf{A}}$ is taken to point outwards from the surface.

2. If the angle between $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ is < 90° then dΦ is positive

If the angle between $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ is > 90° then dΦ is negative



Example: Flux through a spherical surface with a point charge Q at the centre.



1. $\overline{\mathbf{E}}$ is perpendicular to the surface everywhere

   $\Rightarrow$ $\overline{\mathbf{E}}$ and $d\overline{\mathbf{A}}$ are parallel everywhere

   $\Rightarrow$ $\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = EdA$

2. E is the same for all points on the surface (because they are all at the same distance from Q):

$$E = \frac{Q}{4\pi\varepsilon_o r^2}$$

$$\Phi = \int_A \overline{\mathbf{E}}.d\overline{\mathbf{A}} = \int_A EdA = E\int_A dA = \frac{Q}{4\pi\varepsilon_o r^2}\int_A dA$$

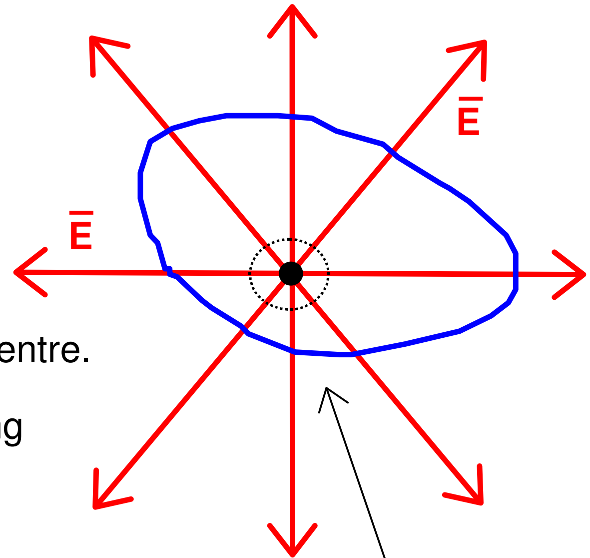
But $\int_S dA$ is just the surface area of the sphere $= 4\pi r^2$.

Therefore $$\Phi = \frac{Q}{\varepsilon_o}$$

Note: 1. Φ is independent of the distance from the charge, r.

2. Φ depends only on Q.

More general example:



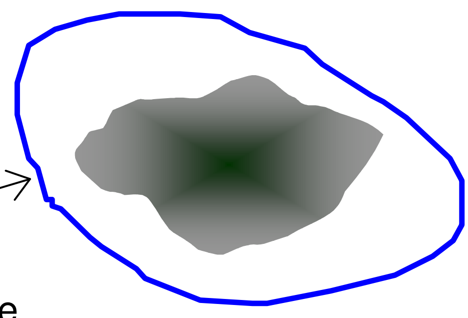
Arbitrary 3-D surface, A, with charge Q somewhere inside

What is Φ through a CLOSED surface of ANY shape due to a point charge Q ANYWHERE inside?

1. Imagine a small sphere with Q at its centre.
2. Any field lines (≡ Electric Flux) passing through this sphere also pass through the surface A.
3. Q/ε_o is the flux through the sphere.
4. Therefore Q/ε_o is also the flux through A.

Even more general example:

What is Φ through an ARBITRARY closed surface containing total charge $Q_{enclosed}$, which is distributed in an ARBITRARY way inside it?



Arbitrary 3-D surface, A, with total charge $Q_{enclosed}$ distributed in some arbitrary way inside

RECALL: THE PRINCIPLE OF SUPERPOSITION

$\bar{\mathbf{E}}$ due to a number of charges Q_i is the vector sum of the $\bar{\mathbf{E}}_i$ due to the individual charges

So: 1. Assume the charge distribution is made up of many small point charges ΔQ_i.

2. Flux through surface from each of these is $\Delta\Phi = \dfrac{\Delta Q_i}{\varepsilon_o}$.

3. The total flux through the surface is the sum of all these contributions:

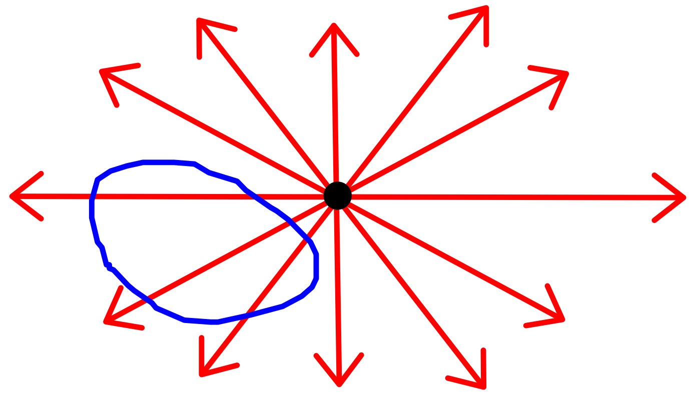
$$\Phi = \sum \frac{\Delta Q_i}{\varepsilon_o} = \frac{1}{\varepsilon_o}\sum \Delta Q_i = \frac{Q_{enclosed}}{\varepsilon_o} .$$

So the answer is still the same as before.

What about the contribution to the flux through a closed surface from a charge OUTSIDE it?

Clearly, any field line (electric flux) that ENTERS the surface at one point must LEAVE it at some other point.

So the total flux through the closed surface from an external charge is ZERO.



CONCLUSION: We have established a very general principle:

If the volume within an arbitrary closed surface contains a total electric charge $Q_{enclosed}$, then the total electric flux through the surface is $Q_{enclosed}/\varepsilon_o$.

or:

$$\Phi = \oint \overline{\mathbf{E}}.d\overline{\mathbf{A}} = \frac{Q_{enclosed}}{\varepsilon_o}$$

This is **GAUSS'S LAW**
MAXWELL'S 1st EQUATION

Another way of expressing it:

The integral of $\overline{\mathbf{E}}$ over a closed surface is equal to the enclosed charge divided by ε_o.

Note: 1. $\int_A$ ⇒ integral over a surface.

$\oint$ ⇒ integral over a CLOSED surface.

2. The surface is not necessarily a real one - we can specify ANY imaginary surface we want when using Gauss's Law.

When to use Gauss' s Law

When you are given some CHARGE DISTRIBUTION and you want to find the ELECTRIC FIELD.

How to use it: a step-by-step procedure:

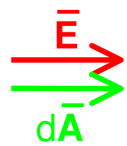
1. Determine the ELECTRIC FIELD PATTERN - draw diagram(s) showing the field lines

2. Choose the best GAUSSIAN SURFACE, to make things simple:

- Inevitably: cylinder, sphere or cube

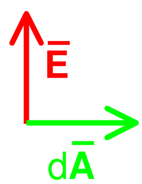
- Try to make $d\overline{\mathbf{A}}$ and $\overline{\mathbf{E}}$ either

PARALLEL:



$$\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = EdA$$

or PERPENDICULAR:



$$\overline{\mathbf{E}} \cdot d\overline{\mathbf{A}} = 0$$

3. Work out the SURFACE INTEGRAL $\Phi = \oint \overline{\mathbf{E}}.d\overline{\mathbf{A}}$.

4. Decide how much charge is INSIDE the surface, $Q_{enclosed}$. Ignore any charge that is outside.

5. Set $\Phi = \dfrac{Q_{enclosed}}{\varepsilon_o}$ and rearrange the equation to find

the magnitude of $\overline{\mathbf{E}}$ as a function of charge and position.

Examples of application of Gauss' s Law

1. Electric field due to a point charge
2. Electric field due to an infinite line of charge
3. Electric field due to an infinite sheet of charge
4. Electric field due to a sphere of uniform charge density

See lecture notes

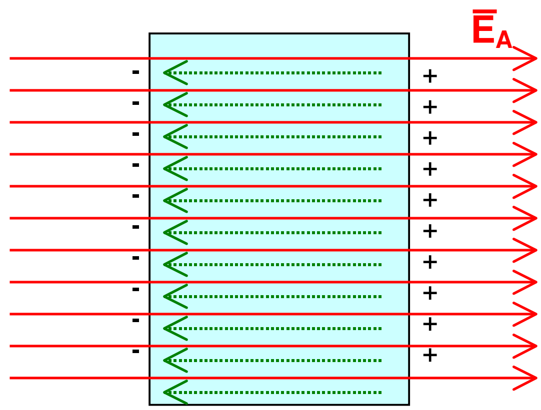
Conductors in electric fields

Using common sense arguments, we can show three important things about how a conductor behaves in the presence of an electric field.

Recall: In a conductor, charges are free to move in response to an electric field.

Consider a piece of conductor placed in an applied electric field $\overline{\mathbf{E}}_A$

Electrons move in a direction opposite to $\overline{\mathbf{E}}_A$



Negative charge builds up on the left
Positive charge is left behind on the right

These INDUCED CHARGES produce their own electric field, $\overline{\mathbf{E}}_i$, which OPPOSES $\overline{\mathbf{E}}_A$.

If $E_i < E_A$: electrons move to the left, making E_i increase
If $E_i > E_A$: electrons move to the right, making E_i decrease

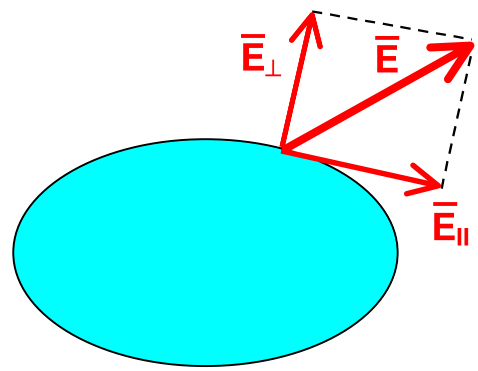
$\Rightarrow$ At equilibrium, $\overline{\mathbf{E}}_i = -\overline{\mathbf{E}}_A \quad \Rightarrow \quad \overline{\mathbf{E}}_{total} = 0.$

So,

1. In equilibrium (ELECTROSTATICS) the electric field inside a perfect conductor is zero.

Using a similar argument, we can show that

2. At the surface of a conductor, the electric field is perpendicular to the surface.



Proof: Imagine that, at some instant, $\bar{\mathbf{E}}$ is NOT perpendicular to the surface.

Resolve $\bar{\mathbf{E}}$ into two components: one parallel to the surface, $\bar{\mathbf{E}}_{||}$, and one perpendicular, $\bar{\mathbf{E}}_{\perp}$.

Clearly, $\bar{\mathbf{E}}_{||}$ will cause charge to move across the surface

→ a separation of positive and negative charges
→ an opposing electric field which exactly cancels out $\bar{\mathbf{E}}_{||}$.

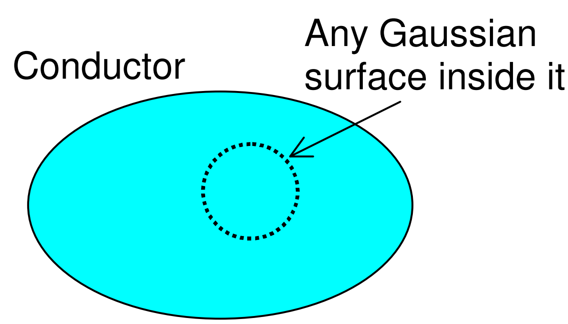
So, in equilibrium, the total component parallel to the surface is zero.

An important consequence of the fact that $\bar{\mathbf{E}}$ is zero inside a perfect conductor is that

3. In a perfect conductor, all excess charge resides at the surface

Proof:

E = 0 at all points on the Gaussian surface



$\Rightarrow \Phi = 0 \Rightarrow Q_{enclosed} = 0.$

$\Rightarrow$ all the excess charge must be on the surface.

Example

1. Electric field above a charged plane conductor

See lecture notes.