Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-122 Mathematical Techniques 2

Time Allowed: 2 hours 30 minutes

Date: May, 2012

Time:

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.**

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. A.J. Martin, xxx

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SECTION A. Attempt answers to all questions.

A1 The Cartesian and polar coordinates of a point in two dimensions are related to each other by: $x = r\cos\theta$ and $y = r\sin\theta$. Find expressions for r, θ, $\frac{\partial x}{\partial r}$, $\frac{\partial y}{\partial r}$, $\frac{\partial x}{\partial \theta}$, $\frac{\partial y}{\partial \theta}$, $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$. **[6]**

A2 Evaluate the integral:

$$\int_{y=0}^{2}\int_{x=0}^{3} (xy^2)\, dxdy.$$

[4]

A3 Write down an expression for the dot product $\vec{u}\cdot\vec{v}$ and vector product, $\vec{u}\times\vec{v}$ of two vectors $\vec{u} = (u_x, u_y, u_z)$, $\vec{v} = (v_x, v_y, v_z)$ in terms of their components. Hence find the cross products of the unit vectors $\hat{\imath}\times\hat{\jmath}$ and $\hat{k}\times\hat{\jmath}$. **[5]**

A4 If r is a scalar field $r = \sqrt{x^2+y^2+z^2}$, calculate the gradient of r and $\nabla^2 r$. **[4]**

A5 Explain what is meant by a conservative vector field. Why is a conservative vector field always irrotational? **[4]**

A6 Explain what is meant by a solenoidal vector field. Why can a solenoidal vector field always be defined as a vector potential? **[4]**

A7 For the matrices:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -2 \\ 2 & 2 \end{pmatrix}.$$

Calculate the determinants $\mid A \mid$ and $\mid B \mid$. Also show that AB is not equal to BA and that $(AB)^T = B^T A^T$. **[5]**

A8 State the conditions that make a matrix (i) *symmetric*, (ii) *orthogonal* and (iii) *unitary*. In each case illustrate your answer with an example. **[6]**

A9 What is the *trace* of a square matrix and what happens to it under a similarity transformation that diagonalises the matrix? **[4]**

A10 Find the eigenvalues and corresponding normalized eigenvectors of the matrix: $\begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$ **[4]**

A11 Solve the differential equation $\frac{dy}{dx} = x + xy$ with the boundary condition $y(0) = 1$. **[4]**

SECTION B. Answer two of the four questions in this section.

B1

(i) Evaluate the double integral:

$$\int\int_R (a + \sqrt{x^2 + y^2})dxdy$$

where R is the region bounded by the circle $x^2 + y^2 = a^2$ **[7]**

(ii) Evaluate the line integral

$$S = \frac{1}{2}\oint_C \vec{r} \times d\vec{r},$$

in moving a particle once anticlockwise around the unit circle C in the $x - y$ plane defined by the equation $x^2 + y^2 = a^2$. **[8]**

(iii) Evaluate the surface integral:

$$I = \int_S \vec{a} \cdot d\vec{S},$$

where $\vec{a} = x\hat{i}$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z > 0$. **[10]**

B2

(i) Show that $\nabla \cdot \phi\vec{A} = (\nabla\phi) \cdot \vec{A} + \phi(\nabla \cdot \vec{A})$ for any vector field $\vec{A}$ and scalar field ϕ. **[7]**

(ii) A scalar field is defined by $\phi(r) = \frac{e^{-\lambda r}}{r}$, where $\vec{r} = (x, y, z)$ and λ is a positive real number. Calculate $\nabla\phi$. **[7]**

(iii) A compressible fluid has time-varying position dependent density $\rho(\vec{r}, t)$ and a velocity field $v(\vec{r}, t)$. Show that, for an arbitrary volume V, in which fluid is neither created or destroyed:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0$$

[11]

B3

(i) Consider the Hermitian matrix, $H = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, where $i^2 = -1$. Find the eigenvalues and eigenvectors. Find the normalized eigenvectors of H. Show that the matrix formed by the normalized eigenvectors as columns is unitary. **[8]**

(ii) Construct an orthonormal set of eigenvectors for the matrix

$$B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

Use your set of eigenvectors to diagonalise the matrix B and verify that the resulting matrix has the eigenvalues of B on its diagonal. **[9]**

(iii) Show the set of simultaneous equations:

$$\begin{aligned} 2x + 4y + 3z &= 4 \\ x - 2y - 2z &= 0 \\ -3x + 3y + 2z &= -7 \end{aligned}$$

has a unique solution and find that solution. **[8]**

B4

(i) Solve the first-order differential equation:

$$\frac{dy}{dx} + 2xy = 4x$$

giving the general solution. Indicate the constant of integration required and show how it is determined by the boundary condition, $\frac{dy(0)}{dx} = 4$. **[7]**

(ii) Solve the linear second order differential equation:

$$\frac{d^2x(t)}{dt^2} + 2\beta\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

giving the general solutions and how they differ depending on the relationship between β and ω_0. Indicate the constants of integration required. **[7]**

(iii) Solve the linear second order differential equation:

$$\frac{d^2y}{dx^2} + 4y = x^2 \sin 2x$$

giving the general solution expressed in terms of sine and cosine functions. Indicate the constants of integration required. **[11]**