Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-122 Mathematical Techniques 2

Time Allowed: 2 hours 30 minutes

Date: 12 May, 2011

Time: 14:30

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.**

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. A.J. Martin, Dr. A. Bevan

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SECTION A. Attempt answers to all questions.

A 1 Find the roots (*i.e.* solutions for z) of the equation $z^2 + 2z + 10 = 0$ and express them in the form $z = a + ib$. How are they related? **[3]**

A 2 The Cartesian and polar coordinates of a point in two dimensions are related to each other by: $x = r\cos\theta$ and $y = r\sin\theta$. Find expressions for r, θ, $\frac{\partial x}{\partial r}$, $\frac{\partial y}{\partial r}$, $\frac{\partial x}{\partial \theta}$, $\frac{\partial y}{\partial \theta}$, $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$. **[6]**

A 3 Evaluate the integral:

$$\int_{y=0}^{2}\int_{x=0}^{3} (x + y^2)\, dxdy.$$

[4]

A 4 Write down an expression for the dot product $\vec{u}\cdot\vec{v}$ and vector product, $\vec{u}\times\vec{v}$ of two vectors $\vec{u} = (u_x, u_y, u_z)$, $\vec{v} = (v_x, v_y, v_z)$ in terms of their components. Hence find the cross products of the unit vectors $\hat{i}\times\hat{j}$ and $\hat{k}\times\hat{j}$. **[5]**

A 5 Evaluate $\nabla\phi(x, y, z)$ for the scalar field $\phi = e^x y^2 z^3$. Also evaluate $\nabla\cdot\vec{A}(x, y, z)$ and $\nabla\times\vec{A}(x, y, z)$ for the vector field $\vec{A} = (x - y, x + y, z)$. **[6]**

A 6 For the matrices:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -2 \\ 2 & 2 \end{pmatrix}.$$

Calculate the determinants $\mid A \mid$ and $\mid B \mid$. Also show the AB is not equal to BA and that $(AB)^T = B^T A^T$. **[5]**

A 7 State the conditions that make a matrix (i) *symmetric*, (ii) *orthogonal* and (iii) *unitary*. In each case illustrate your answer with an example. **[6]**

A 8 What is the *trace* of a square matrix and what happens to it under a similarity transformation that diagonalises the matrix? **[4]**

A 9 For an $n \times n$ square matrix, write down the characteristic equation obeyed by the eigenvalues. **[3]**

A 10 Show that $\vec{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvalue of the matrix matrix: $\begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$ and find the corresponding eigenvalue. **[4]**

A 11 Solve the differential equation $\frac{dy}{dx} = e^x + 1$ with the boundary condition $y(0) = 0$. **[4]**

SECTION B. Answer two of the four questions in this section.

B1

(i) Evaluate the double integral:

$$\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} \frac{dxdy}{(x^2+y^2+a^2)^2}.$$

[7]

(ii) Find the work done:

$$W = \oint \vec{F}\cdot d\vec{r},$$

in moving a particle once anticlockwise around the unit circle C in the $x-y$ plane defined by the equation $x^2+y^2=16$, if the field is given by $\vec{F}=(x-y,x+y)$. **[8]**

(iii) Evaluate the surface integral:

$$I = \int_S \vec{a}\cdot d\vec{S},$$

where $\vec{a}=x\hat{i}$ and S is the surface of the hemisphere $x^2+y^2+z^2=a^2$ with $z>0$.
[10]

B2

(i) Explain what is meant by a conservative vector field? Why is a conservative vector field always irrotational? **[4]**

(ii) Explain what is meant by a solenoidal vector field? Why can a solenoidal vector field always be defined as a vector potential? **[4]**

(iii) If r is a scalar field $r=\sqrt{x^2+y^2+z^2}$, calculate the gradient of r and $\nabla^2 r$. **[6]**

(iv) Evaluate the line integral:

$$I = \int_A^B \vec{a}\cdot d\vec{r},$$

where $\vec{a}=(xy^2+z)\hat{i}+(x^2y+2)\hat{j}+x\hat{k}$. A is the point (c,c,h) and B is the point $(2c,c/2,h)$ along the two different paths:

$$C1 \quad \text{given} \quad \text{by} \quad x=cu, y=c/u, z=h \quad \text{and}$$

$$C2 \quad \text{given} \quad \text{by} \quad 2y=3c-x, z=h.$$

Show that the vector field $\vec{a}$ is in fact conservative. **[11]**

B3

(i) Consider the Hermitian matrix, $H = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, where $i^2 = -1$. Find the eigenvalues and eigenvectors. Find the normalized eigenvectors of H. Show that the matrix formed by the normalized eigenvectors is unitary. **[8]**

(ii) Consider the square matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}.$$

Calculate the determinant of A and the inverse matrix A^{-1}. Show by direct matrix multiplication that AA^{-1} is equal to the identity matrix. **[8]**

(iii) Construct an orthonormal set of eigenvectors for the matrix

$$B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

Use your set of eigenvectors to diagonalise the matrix B and verify that the resulting matrix has the eigenvalues of B on its diagonal. **[9]**

B4

(i) Solve the first-order differential equation:

$$\frac{dy}{dx} + 2xy = 4x$$

giving the general solution. Indicate the constant of integration required and show how it is determined by the boundary condition, $\frac{dy(0)}{dx} = 4$. **[8]**

(ii) Solve the linear second order differential equation:

$$\frac{d^2x(t)}{dt^2} + \omega_0^2 x(t) = 0$$

giving the general solution. Indicate the two constants of integration required and show how they are determined by the boundary conditions $x(0) = 0$ and $\frac{dx(0)}{dt} = \omega_0$. Give the resulting solution for $x(t)$. **[8]**

(iii) Solve the linear second order differential equation:

$$\frac{d^2x(t)}{dt^2} + 2\beta\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

giving the general solutions and how they differ depending on the relationship between β and ω_0. Indicate the constants of integration required. **[9]**

End of paper