Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-122 Mathematical Techniques 2

Time Allowed: 2 hours 30 minutes

Date: 14^{th} May, 2010

Time: 10:00 - 12:30

Instructions: **Answer ALL questions in section A. Answer ONLY TWO questions from section B. Section A carries 50 marks, each question in section B carries 25 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question.**

Numeric calculators are permitted in this examination. Please state on your answer book the name and type of machine used. Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important Note: The academic regulations state that possession of unauthorised material at any time when a student is under examination conditions is an assessment offence and can lead to expulsion from the college. Please check now to ensure that you do not have any notes in your possession. If you have any then please raise your hand and give them to an invigilator immediately. Exam papers cannot be removed from the exam room

You are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Examiners: Dr. A.J. Martin, Dr. A. Bevan

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SECTION A. Attempt answers to all questions.

A 1 Draw an Argand diagram and mark the complex number $z = a + ib$ on it. If z is expressed in polar form $z = re^{i\theta}$, give expressions for a and b in terms of r and θ. **[4]**

A 2 Write down all the cube roots of $e^{i\pi/2}$ in the form $a+ib$ and also in the form $re^{i\theta}$. **[4]**

A 3 Write down an expression for the scalar product, $\vec{u} \cdot \vec{v}$ of two vectors $\vec{u} = (u_x, u_y, u_z)$, $\vec{v} = (v_x, v_y, v_z)$ in terms of their components. **[4]**

A 4 Write down an expression for the vector product, $\vec{u} \times \vec{v}$ of two vectors $\vec{u} = (u_x, u_y, u_z)$, $\vec{v} = (v_x, v_y, v_z)$ in terms of their components. Hence find the cross products of the unit vectors $\vec{i} \times \vec{j}$, $\vec{j} \times \vec{k}$ and $\vec{i} \times \vec{k}$. **[6]**

A 5 Evaluate the integral:

$$\int_{x=0}^{3} \int_{y=0}^{4} x^2 y \, dy dx$$

[5]

A 6 For the matrices:

$$P = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

Calculate the determinants $\mid P \mid$ and $\mid Q \mid$. Also show the PQ is not equal to QP and that $P^T Q^T = (QP)^T$. **[6]**

A 7 Explain what is meant by a *symmetric* matrix, an *orthogonal* matrix, a *unitary* matrix and a *Hermitian* matrix. Give (2×2) matrix examples of each. **[6]**

A 8 Find the eigenvalues of the matrix: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ **[5]**

A 9 Write down a Fourier series for a function $f(x)$ which is periodic with period 2π for all x. Write down expressions for the coefficients a_k and b_k for $f(x)$ in terms of the basic functions $\cos kx$ and $\sin kx$, with $k = 0, 1, 2, \ldots$. **[5]**

A 10 Solve the differential equation $\frac{dy}{dx} = x - 3$ with the boundary condition $y(0) = 0$. **[5]**

SECTION B. Answer two of the four questions in this section.

B1

(i) A complex number is given by $z = a + ib$. Express $1/(z+i)$ and $1/z^2$ in the form $x + iy$ where x and y are real numbers. **[7]**

(ii) Find

$$\text{grad} \ \frac{e^{\lambda r}}{r}$$

where r is the modulus of the position vector $\vec{r} = (x, y, z)$ and λ is a positive real number. **[8]**

(iii) Using polar coordinates show that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

[10]

B2

(i) Prove that div curl $\vec{A}(x, y, z) = 0$ for any vector field $\vec{A}(x, y, z)$. **[7]**

(ii) Using the identity $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$, find expressions for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$ in terms $\cos(\theta_1)$, $\cos(\theta_2)$ $\sin(\theta_1)$ and $\sin(\theta_2)$. **[8]**

(iii) Find the work done,

$$W = \oint \vec{F} \cdot d\vec{r}$$

in moving a particle once anticlockwise around the unit circle C in the $x-y$ plane define by the equation $x^2+y^2 = 1$, if the force field is given by $\vec{F} = (x-3y, 2x+y)$.

[10]

B3

(i) Solve the linear first-order differential equation for radioactive decay

$$\frac{dN(t)}{dt} = -\kappa N(t),$$

giving the general solution. Indicate the constant of integration required and show how it is determined by the boundary condition, $N(t=0) = N_0$ **[7]**

(ii) A function $f(x)$, which is periodic in x with a period of 2π, is given by:

$$f(x) = \{ \begin{array}{ll} 1 & \text{for} \quad 0 \leq x \leq \pi \\ 0 & \text{for} \quad \pi \leq x \leq 2\pi \end{array}$$

sketch the function, and calculate the Fourier coefficients a_k and b_k of $f(x)$ for $k = 0, 1, 2, 3, 4$. **[9]**

(iii) Consider the square matrix

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculate the inverse matrix A^{-1}. Show by direct matrix multiplication that AA^{-1} is equal to the identity matrix. **[9]**

B4

(i) Consider the Hermitian matrix $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, where $i^2 = -1$. Find the eigenvalues, the determinant and the trace of H and show how they are related. Find the normalized eigenvalues of H. **[8]**

(ii) Find the Fourier Transform $F(k)$ of the Gaussian

$$f(x) = \frac{1}{\sqrt{\sigma}} e^{-x^2/2\sigma^2}$$

You may use the integral $\int_{-\infty}^{\infty} e^{-ax^2} e^{bx} dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$. Sketch the functions $f(x)$ and $F(k)$ to illustrate their behaviour for large σ. **[8]**

(iii) Solve the linear second order differential equation

$$\frac{d^2x(t)}{dt^2} + 6\frac{dx(t)}{dt} + 25x(t) = 0$$

giving the general solution. Indicate the two constants of integration required for a real solution. **[9]**