9.1) $\mu_A = \begin{pmatrix} 1.0 \\ 2.0 \end{pmatrix}$ $V_A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$

$\mu_B = \begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix}$ $V_B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$\Delta_\mu = \mu_A - \mu_B = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix}$

$W = V_A + V_B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

$\det(W) = 8 - 9 = -1$

$\therefore W^{-1} = -\begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$

$\therefore \alpha \propto W^{-1} \Delta_\mu$

$\propto \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\propto \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

9.2 $\mu_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $V_A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\therefore W = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$\mu_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $V_B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\det(W) = 8$

$\Delta_\mu = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore W^{-1} = \frac{1}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$

$\Rightarrow \alpha \propto W^{-1}\Delta_\mu$

$\propto \frac{1}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\propto \frac{1}{8}\begin{pmatrix} 4 \\ -4 \end{pmatrix}$

$\propto \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

9.3) $\mu_A = \begin{pmatrix} 0.2 \\ 0.7 \\ 0.1 \end{pmatrix}$ $V_A = \begin{pmatrix} 0.1 & 0.0 & 0.0 \\ 0.0 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.2 \end{pmatrix}$

$\mu_B = \begin{pmatrix} 0.6 \\ 1.5 \\ 0.2 \end{pmatrix}$ $V_B = \begin{pmatrix} 0.15 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 \\ 0.0 & 0.0 & 0.3 \end{pmatrix}$

$\Delta\mu = \begin{pmatrix} -0.4 \\ -0.8 \\ -0.1 \end{pmatrix}$ $\omega = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$

$\det(\omega) = \frac{1}{20}$

$C(\omega) = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.125 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$

$\omega^{-1} = \frac{C(\omega)}{\det(\omega)} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$\therefore \alpha \propto \omega^{-1}\Delta\mu = \begin{pmatrix} -1.6 \\ -2.0 \\ -0.2 \end{pmatrix}$

9.4)

$$\mu_A = \begin{pmatrix} 1.0 \\ 0.7 \\ 0.5 \end{pmatrix} \qquad V_A = \begin{pmatrix} 0.2 & 0.1 & 0.0 \\ 0.1 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.2 \end{pmatrix}$$

$$\mu_B = \begin{pmatrix} 0.6 \\ 1.0 \\ 0.2 \end{pmatrix} \qquad V_B = \begin{pmatrix} 0.15 & 0.0 & 0.1 \\ 0.0 & 0.3 & 0.0 \\ 0.1 & 0.0 & 0.3 \end{pmatrix}$$

$$\Delta\mu = \begin{pmatrix} 0.4 \\ -0.3 \\ 0.3 \end{pmatrix} \qquad W = \begin{pmatrix} 0.35 & 0.1 & 0.1 \\ 0.1 & 0.4 & 0.0 \\ 0.1 & 0.0 & 0.5 \end{pmatrix}$$

$$\det(W) = 0.35 \begin{vmatrix} 0.4 & 0 \\ 0 & 0.5 \end{vmatrix} - 0.1 \begin{vmatrix} 0.1 & 0 \\ 0.1 & 0.5 \end{vmatrix} + 0.1 \begin{vmatrix} 0.1 & 0.4 \\ 0.1 & 0 \end{vmatrix}$$

$$= 0.35 \times 0.4 \times 0.5 - 0.1 \times 0.1 \times 0.5 + 0.1 \times (-1) \times 0.1 \times 0.4$$

$$= 0.061$$

$$\therefore W^{-1} = \frac{1}{0.061} \begin{pmatrix} 0.20 & -0.05 & -0.04 \\ -0.05 & 0.17 & 0.01 \\ -0.04 & 0.01 & 0.13 \end{pmatrix}$$

$$= \begin{pmatrix} 3.2787 & -0.8197 & -0.6557 \\ -0.8197 & 2.705 & 0.1639 \\ -0.6557 & 0.1639 & 2.1311 \end{pmatrix}$$

$$\therefore \alpha \propto W^{-1} \Delta\mu = \begin{pmatrix} 1.3607 \\ -1.0901 \\ 0.3279 \end{pmatrix} \quad \text{(4dp)}$$

SDA 9 – MUA

9.5)

P/E	P_e	P_μ	P_π	classification
0.0	0.0	1.995	0.0876	μ
0.1	0.0	1.760	0.2700	μ
0.2	0.0007	1.210	0.6476	μ
0.3	0.0044	0.6476	1.210	π
0.4	0.0222	0.2700	1.7603	π
0.5	0.0876	0.0876	1.9947	π
0.6	0.2700	0.0222	1.7603	π
0.7	0.6476	0.0044	1.210	π
0.8	1.210	0.0007	0.6476	e
0.9	1.760	0.0	0.2700	e
1.0	1.995	0.0	0.0876	e

Model

$e:\ G(x;\ 1.0, 0.2)$

$\mu:\ G(x;\ 0.0, 0.2)$

$\pi:\ G(x;\ 0.5, 0.2)$

Use Bayesian classifier
→ majority rule classification of each event based on the hypothetical models.

9.6) $\Omega = \{-3, -2, 0, 1, 4, 5\}$

~~MODEL~~

$S = G(\cancel{x};\ 0, 1)$

$B =$ Uniform.

ω_i	$P(\text{sig})$	$P(\text{bg})$	classification
−3	0.0044	1/6	B
−2	0.0540	1/6	B
0	0.3989	1/6	S
1	0.2420	1/6	S
4	0.0001	1/6	B
5	0.0000	$1/6\ (= 0.1\dot{6})$	B

9.7)

i	x	y	P_A	P_B	Classification
1	−1	1	0.0585	0.0131	A
2	0	2	0.0215	0.0965	B
3	1	1	0.0585	0.0965	B
4	3	3	0.0000	0.0131	B
5	2	−1	0.0131	0.0011	A
6	0.5	1.0	0.0852	0.0852	A or B (equal ranking)

$P_A(x) = G(x; 0, 1)$ $P_B(x) = G(x; 1, 1)$

$P_A(y) = G(y; 0, 1)$ $P_B(y) = G(y; 2, 1)$

9.8) $P_A(x) = P_A(y) =$ uniform $P_B(x)$ & $P_B(y)$ as per previous question.

i	x	y	P_A	P_B	Classification
1	−1	1	$(1/6)^2$	0.0131	A
2	0	2	$(1/6)^2$	0.0965	B
3	1	1	$(1/6)^2$	0.0965	B
4	3	3	$(1/6)^2$	0.0131	A
5	2	−1	$(1/6)^2$	0.0011	A
6	0.5	1.0	$(1/6)^2$	0.0852	B

$(1/6)^2 = 0.02\dot{7}$

9.9) $P_A = \frac{3}{4}(2x - x^2)$ $\quad N_A = 10 \quad x \in [0, 2]$

$P_B = 1/2 \quad N_B = 20$

Optimise a cut on x to maximise N_A / N_B in the interval $[0, X_c]$

$$R = \frac{N_A \int_0^{X_c} P_A \, dx}{N_B \int_0^{X_c} P_B \, dx}$$

\# is the test statistic $\frac{N_A}{N_B}$ as a function of X_c

$$N_B \int_0^{X_c} P_B \, dx = 20 \int_0^{X_c} 1/2 \, dx = 10 X_c \quad \text{——— (1)}$$

$$N_A \int_0^{X_c} P_A \, dx = \frac{30}{4} \int_0^{X_c} 2x - x^2 \, dx$$

$$= \frac{15}{2} \left[x^2 - \frac{x^3}{3} \right]_0^{X_c}$$

$$= \frac{15}{2} \left[X_c^2 - \frac{X_c^3}{3} \right]$$

$$= \frac{15 X_c^2}{2} - \frac{5 X_c^3}{2} \quad \text{——— (2)}$$

$$\frac{(2)}{(1)} = R(X_c) = \frac{3}{4} X_c - \frac{X_c^2}{4}$$

The optimal cut is obtained for $R'(X_c) = 0$ where:

$$\frac{3}{4} - \frac{X_c}{2} = 0$$

$$\Rightarrow X_c = 3/2$$

\# Note $\frac{N_A}{N_B}$ is a scale factor that does not alter the optimal cut value as can be seen from the calculation of $R(X_c)$ above.

9.10) The yields multiply the ratio of integrals, and as noted do not play a role in the exact value of the optimal cut on x.

9.11) $S = \dfrac{(\mu_A - \mu_B)^2}{\sigma_A^2 + \sigma_B^2}$

	A	B
μ	1	3
σ	2	1

$\Rightarrow S = 0.8$

9.12)

	A	B
μ	1	0
σ	1	1

$\Rightarrow S = 0.5$

9.13) $P_A = 6(x - x^2)$ yield for signal : N_A

$P_B = 1$ yield for background : N_B

$x \in [0,1]$

$$R(X_c) = \frac{\frac{N_A}{N_B} \cdot 6 \int_0^{X_c} x - x^2 \, dx}{\int_0^{X_c} dx}$$

$$= \frac{6 N_A}{X_c N_B} \cdot \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^{X_c}$$

$$= \frac{N_A}{N_B} \left[3X_c - 2X_c^2 \right]$$

$R' = 0$ gives the optimal cut position

$$R'(X_c) = \frac{N_A}{N_B} \left[3 - 4X_c \right]$$

$\therefore \underline{\underline{X_c = 3/4}}$

9.14) $P_A = 12(x^2 - x^3)$ $x \in [0,1]$

$P_B = 1$ N_A, N_B undefined

optimise on S/B in $[0, X_c]$

$$\therefore R(X_c) = \frac{N_A}{N_B} \cdot \frac{12\int_0^{X_c} x^2 - x^3\, dx}{\int_0^{X_c} dx}$$

$$= \frac{12 N_A}{X_c N_B} \int_0^{X_c} x^2 - x^3\, dx$$

$$= \frac{12 N_A}{X_c N_B}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{X_c}$$

$$= \frac{N_A}{N_B} \cdot \frac{1}{X_c}\left(4X_c^3 - 3X_c^4\right)$$

$$= \frac{N_A}{N_B}\left(4X_c^2 - 3X_c^3\right)$$

As $R' = 0$ gives the optimal cut value for X_c, and N_A/N_B is just a scale factor

→ drop the scale:

$$\therefore \frac{dR}{dX_c} \propto 8X_c - 9X_c^2$$

$\therefore$ Solve $8X_c = 9X_c^2$ for cut value

$$\Rightarrow X_c = 8/9 = 0.\dot{8}$$

9.15) All of these problems have a peaking signal component.

$\therefore$ Should simultaneously optimise upper and lower cut values to improve S/B. i.e. determine $[X_L, X_H]$ that maximises S/B.

In general this can become difficult, so often one numerically solves such a problem.