8.1) It may be useful to fit data
i) in order to extract the values of one or more parameters (with uncertainties)
ii) to obtain a (multi-dimensional) confidence interval
iii) Average results via a simultaneous fit.

8.2) ν = number of degrees of freedom = number of data minus the number of constraints; typically (i.e. without additional constraint) $\nu = N-1$; as the total number of data is itself a constraint.

8.3) $x_1 = 1.2 \pm 0.3$ χ^2 scan from 1.0 to 2.0 in steps of 0.1
$x_2 = 1.8 \pm 0.3$

x	$\chi_1^2 = \left(\frac{x_1 - x}{\sigma_1}\right)^2$	$\chi_2^2 = \left(\frac{x_2 - x}{\sigma_2}\right)^2$	χ^2_{TOT}
1.0	0.444	$7.\dot{1}$	$7.\dot{5}$
1.1	0.111	$5.\dot{4}$	$5.\dot{5}$
1.2	0	4	4.0
1.3	0.111	$2.\dot{7}$	$2.\dot{8}$
1.4	0.444	$1.\dot{7}$	$2.\dot{2}$
1.5	1.0	1	2
1.6	$1.\dot{7}$	$0.\dot{4}$	$2.\dot{2}$
1.7	$2.\dot{7}$	$0.\dot{1}$	$2.\dot{8}$
1.8	4.0	0.0	4.0
1.9	$5.\dot{4}$	$0.\dot{1}$	$5.\dot{5}$
2.0	$7.\dot{1}$	$0.\dot{4}$	$7.\dot{5}$

$\chi^2_{min} = 2$, at $x = 1.5$
$\Delta\chi^2 = 1$ is approximately ± 0.2 i.e est. $x = 1.5 \pm 0.2$

c.f. result of Question 5.8 where $x = 1.5 \pm 0.2$. One gets the same result at this level of precision.

N.B. Could use the expectation that the χ^2 is parabolic about the minimum to infer $\sigma_x \sim 0.2(21)$, but one would round the result to the quoted level of precision.

8.4) $y = ax^2 + b$

$\therefore \chi^2 = \sum_i \left(\frac{y_i - a x_i^2 - b}{\sigma_i} \right)^2$ Assume $\sigma_i = \sigma$ (same $\forall\, i$)

$\therefore$ Determine turning points via $\partial \chi^2 / \partial a = 0$ and $\partial \chi^2 / \partial b = 0$

$$\frac{\partial \chi^2}{\partial a} = \frac{1}{\sigma} \sum_i \frac{\partial}{\partial a} \left(y_i - a x^2 - b \right)^2$$

$$= \frac{2}{\sigma} \sum_i \left(y_i - a x_i^2 - b \right) \cdot \left(- x_i^2 \right)$$

$$= \frac{2}{\sigma} \sum_i - y_i x_i^2 + a x_i^4 + b x_i^2$$

$$= \frac{2N}{\sigma} \left[a \overline{x^4} + b \overline{x^2} - \overline{y x^2} \right]$$

$$\frac{\partial \chi^2}{\partial b} = \frac{1}{\sigma} \sum_i \frac{\partial}{\partial b} \left(y_i - a x_i^2 - b \right)^2$$

$$= \frac{2}{\sigma} \sum_i \left(- y_i + a x_i^2 + b \right)$$

$$= \frac{2N}{\sigma} \left(a \overline{x^2} + b - \overline{y} \right)$$

$\therefore \quad a \overline{x^4} + b \overline{x^2} - \overline{y x^2} = 0$ ——— ①

and $\quad a \overline{x^2} + b - \overline{y} = 0$ ——— ②

② $\Rightarrow b = \overline{y} - a \overline{x^2}$

$\therefore \quad a \overline{x^4} + \left(\overline{y} - a \overline{x^2} \right) \overline{x^2} - \overline{y x^2} = 0$

$\therefore \quad a \left[\overline{x^4} - \left(\overline{x^2} \right)^2 \right] = \overline{y x^2} - \overline{y}\, \overline{x^2}$

$$\boxed{\begin{aligned} \therefore \quad a &= \frac{\overline{y x^2} - \overline{y}\, \overline{x^2}}{\overline{x^4} - \left(\overline{x^2} \right)^2} \\ b &= \overline{y} - a \overline{x^2} \end{aligned}}$$

8.5) Least squares for $y = ax + b$ where

$$a = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2} \qquad b = \bar{y} - a\bar{x}$$

$$\sigma_a^2 = \frac{\sigma^2}{N(\overline{x^2} - \bar{x}^2)} \quad \text{and} \quad \sigma_b^2 = \frac{\sigma^2\,\overline{x^2}}{N(\overline{x^2} - \bar{x}^2)}$$

where $\sigma_y = 0.1$

x	y	x^2	y^2	xy
1.1	2.0	1.21	4	2.2
1.5	2.9	2.25	8.41	4.35
2.0	4.2	4.0	~~17~~ .64	8.4
3.1	6.0	9.61	36	18.6
4.2	8.0	17.64	64	33.6
5.0	10.0	25	100	50

$\therefore \langle x \rangle = 2.81\dot{6}$ $\qquad N = 6$

$\langle y \rangle = 5.51\dot{6}$

$\langle x^2 \rangle = 9.951\dot{6}$

~~$\langle y^2 \rangle = 38.341\dot{6}$~~ (don't need)

$\langle xy \rangle = 19.525$

$\therefore a = 1.975 \pm 0.029$

$\therefore b = -0.047 \pm 0.091$

8.6) $s_1 = 0.655 \pm 0.024$, $s_2 = 0.59 \pm 0.08$ $s_3 = 0.789 \pm 0.071$

s	χ^2_1	χ^2_2	χ^2_3	χ^2_{TOT}
0.60	5.25	0.016	7.086	12.35
0.61	3.51	0.063	6.356	9.93
0.62	2.13	0.141	5.666	7.93
0.63	1.09	0.25	5.015	6.35
0.64	0.39	0.390	4.404	5.18
0.65	0.043	0.5625	3.832	4.43
0.66	0.043	0.7656	3.301	4.11
0.67	0.39	1	2.809	4.[illegible]
0.68	1.09	1.2656	2.357	4.707
0.69	2.12	1.5625	1.944	5.633
0.70	3.52	1.8906	1.571	6.78

$\leftarrow \chi^2_{min}$; estimate $\Delta\chi^2 = 1$ for $\pm 1\sigma$ errors.

$\boxed{\therefore \text{est } S = 0.66^{+0.025}_{-0.020}}$

8.8) $-\ln \mathcal{L} = 1.1(x-1.3)^2$

what is the best fit result for x?

$x = 1.3$ as $-\ln \mathcal{L}$ is at the minimum for this value.

A change in $+\frac{1}{2}$ from this minimum value corresponds to $\pm 1\sigma$.

$\therefore$ solve

$$0.5 = 1.1(x-1.3)^2 \quad \text{to determine } \pm 1\sigma \text{ bound.}$$

$$\therefore x - 1.3 = \sqrt{\frac{0.5}{1.1}}$$

$$\therefore x = 1.3 \pm \sqrt{\frac{0.5}{1.1}} \quad \text{gives the error soln.}$$

$$\therefore x = 1.30 \pm 0.67$$

8.9) $x_1 = 1.3 \pm 0.1$

$x_2 = 1.1 \pm 0.2$

x	χ_1^2	χ_2^2	χ^2_{tot}	$\Delta\chi^2$
1.1	4	0	4	3.1875
1.15	2.25	0.0625	2.315	1.5
1.2	1	0.25	1.25	0.4375
1.25	0.25	0.5625	0.8125	0
1.3	0	1	1	0.1875
1.35	0.25	1.5625	1.8125	1
1.4	1	2.25	3.25	2.4375

$$\therefore \langle x \rangle = 1.25 {}^{+0.10}_{-0.08}$$

$\chi^2 = 0.8125$ for $\nu = 1$; $P(\chi^2, \nu) = 0.37$.

8.9) $-\ln \Delta \mathcal{L} = 0.5(x+2)^2$; what is x?

Best fit value $= -2.0$

$\pm 1\sigma$ given by $-\ln \Delta \mathcal{L} = \frac{1}{2}$

$\therefore \quad \frac{1}{2} = 0.5\,(x+2)^2$

$\therefore \quad x = -2 \pm 1$

8.10) $-\ln \Delta \mathcal{L} = (x-1)^2$; what is x?

central value of x is $+1$.

change of $\frac{1}{2}$ in $-\ln \Delta \mathcal{L}$ corresponds to $\pm 1\sigma$.

$\Rightarrow \pm 1\sigma \equiv \pm 1/\sqrt{2}$

8.11) $F = -kx \quad \therefore \quad x = -F/k$

$$\therefore \chi^2 = \sum_{i=1}^{N} \left(\frac{x_i - F_i/k}{\sigma_i} \right)^2$$

The best fit value occurs at $\partial\chi^2/\partial k = 0$; assume all σ_i are the same.

$$\frac{\partial \chi^2}{\partial k} = \frac{2}{\sigma^2} \sum_i \frac{\partial}{\partial k} (x_i - F_i/k)^2$$

$$= \frac{2}{\sigma^2} \sum_i \left(x - \frac{F}{k}\right)\left(-\frac{F}{k^2}\right)$$

$$= \frac{2N}{\sigma^2 k^2} \left[\overline{xF} - \frac{\overline{F^2}}{k} \right]$$

$\therefore$ The best fit value for k is given by $k = \dfrac{\overline{F^2}}{\overline{xF}}$

Given

x	0.4	1.1	1.5	2.1	2.5	(cm)
F	1	2	3	4	5	(N)

one finds $k = 1.96$ (N/cm)

8.12) Ohm's law : $V = IR$; I is measured precisely ; given data, determine R using LSR : assume $\sigma_v(I)$ is constant

$$\chi^2 = \sum_i \left(\frac{V_i - I_i R}{\sigma_V^2} \right)^2$$

$$\frac{\partial \chi^2}{\partial R} = \frac{2}{\sigma_v^2} \sum_{i=1}^{N} (-I)(V_i - I_i R)$$

$$= -\frac{2}{\sigma_v^2} \sum_{i=1}^{N} (I_i V_i - I_i^2 R)$$

$$= -\frac{2}{\sigma_v^2} \left(\overline{VI} - \overline{I^2} R \right)$$

$$\therefore \quad R = \frac{\overline{VI}}{\overline{I^2}}$$ gives the best fit value