6.1)

CL(%)	Interval
50	$x \in [-0.67\sigma, +0.67\sigma]$
75	$x \in [-1.15\sigma, +1.15\sigma]$
90	$x \in [-1.64\sigma, +1.64\sigma]$
99	$x \in [-2.58\sigma, +2.58\sigma]$

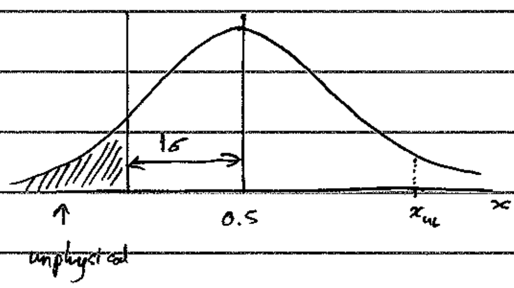
6.2) $G(x;\ 1.0,\ 1.5)$ = PDF.

90% C.L. UL @ a z value of $+1.28$

$\therefore$ 90% CL UL in terms of x is $\mu + 1.28\sigma = 1 + 1.28 \times 1.5 = \underline{\underline{2.92}}$

6.3) $x = 0.5 \pm 0.5$; x is physical for values ≥ 0.

Determine 90% C.L. upper limit on x.



$$\frac{\int_0^{x_{ul}} \text{Pdf}\, dx}{\int_0^{\infty} \text{Pdf}\, dx} = 0.9$$ allows us to determine the value of x_{ul}.

$$\int_{1\sigma}^{\infty} G(x;\ 0.5,\ 0.5)\, dx = 1 - 0.8413 = 0.1587 \quad \therefore \quad \int_0^{\infty} G(x, 0.5, 0.5)\, dx = 0.8413$$

$\therefore$ 90% C.L. UL of the physically allowed space corresponds to $\int_0^{x_{ul}} G(x, 0.5, 0.5)\, dx = 0.75717$

$$\therefore \int_0^{x_{ul}} G(x;\ 0.5, 0.5)\, dx = \int_{-1\sigma}^{\mu} G\, dx + \int_{\mu}^{x_{ul}} G\, dx$$

$$= 0.3413 + 0.41587$$

$\Rightarrow \underline{\underline{x_{ul} = 1.19}}$ (as $z_{ul} = +1.38$)

6.4) $A \to X$

Observe 1 event

What is 95% CL. UL on event yield? → Poisson problem

$\lambda = 1$ $P(r, \lambda) = \frac{\lambda^r e^{-\lambda}}{r!}$

r	$P(r,\lambda)$	running sum
0	0.3679	0.3679
1	0.3679	0.7358
2	0.1839	0.9197
3	0.0613	0.9810

So possible answer: 95% CL UL is 3 events; This limit over-covers and actually has a coverage of 98.1%

Ignoring the discrete nature of the problem one can linearly interpolate between $r=2$ & $r=3$ to obtain an estimate of 2.49 events. Coverage won't be exact but it will be a bit better than the $x_{UL} = 3$ soln.

6.5) $x = 1.0 \pm 0.7$

90% CL. UL @ a z value of $+1.28$.

$\Rightarrow x_{UL} = 1.0 + 1.28 \times 0.7 = 1.896$ (STAT ONLY)

If we include a Gaussian syst. error then $\sigma_x^2 = 0.7^2 + 0.4^2 \Rightarrow \sigma_x = 0.806$

$\Rightarrow x_{UL} = 1.0 + 1.28 \times 0.806 = 2.03$ (STAT + SYST)

Δx_{UL} FROM THE SYST ERROR $= 0.136$

6.6) If 3 events are observed, estimate the corresponding value of λ for the underlying Poisson process.

$$\therefore \quad P(r,\lambda) = \frac{\lambda^3 e^{-\lambda}}{3!} = \frac{\lambda^3 e^{-\lambda}}{6}$$

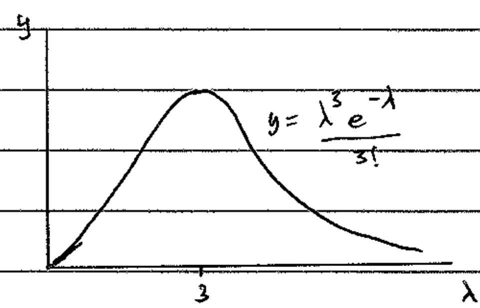
MAXIMUM VALUE IS GIVEN BY THE TURNING POINT OF THIS DISTRIBUTION.

$$\frac{\partial P(3,\lambda)}{\partial \lambda} = \frac{1}{6}\frac{\partial}{\partial \lambda}\left(\lambda^3 e^{-\lambda}\right)$$

$$= \frac{1}{2}\lambda^2 e^{-\lambda} - \frac{1}{6} e^{-\lambda}\lambda^3$$

MPV @ $\frac{\partial P}{\partial \lambda} = 0 \Rightarrow \frac{1}{2}\lambda^2 e^{-\lambda} = \frac{1}{6} e^{-\lambda}\lambda^3$

$\therefore \quad \lambda = 3$ (as expected from the data)



From 2D poisson curve in notes; estimate $\lambda \in [1.6, 5.1]$ @ 68.3% C.L.

6.7) Compare the 95% C.L. upper limit on λ with the upper 68.3% CL. bound from the two sided interval studied in Q 6.6.

Assume $\lambda = 3 \Rightarrow$ 95% CL UL is approx 7.8 as read off from the figure.

@ 95% CL; one obtains a one sided interval that contains $\lambda \pm 1\sigma$; ergo should place a limit on λ for this problem, rather than just reporting the central value & estimated error on λ.

6.8) ~~What is the probability of observing three or less heads when tossing coin~~

$p = q = 0.5$ $n = 10$; find 90% CL UL on r.

$$P(r; n\, p) = p^r (1-p)^{n-r} \frac{n!}{r!\,(n-r)!}$$

$$\sum_{r=i}^{r_{UL}} P(r; n, p) = 0.9$$ is desired ; discrete problem so we must choose between under or over coverage.

$r \leq 7$ @ 94.53 % CL. is the result giving overcoverage

$r \leq 6$ @ 82.81 % CL. " " " undercoverage

Linearly interpolating one can estimate $r \sim 6.6$ gives the 90% CL.

6.9) Binomial problem : $n = 10$

$r = 0$

$p = ?$

Determine the 90 % CL UL on p!

$$P(0; 10, p) = p^0 (1-p)^{10} \frac{10!}{10!\,0!}$$

$$= (1-p)^{10}$$

$\therefore$ want to find P_{UL} where

$$\int_0^{P_{UL}} (1-p)^{10}\, dp = 0.9 \int_0^1 (1-p)^{10}\, dp$$

$$\therefore \left[\frac{-1}{11} (1-p)^{11} \right]_0^{P_{UL}} = 0.9 \left[\frac{-1}{11} (1-p)^{11} \right]_0^1$$

$$\therefore (1 - P_{UL})^{11} - 1 = -0.9$$

$$\therefore (1 - P_{UL})^{11} = 0.1$$

$$\therefore P_{UL} = 1 - \sqrt[11]{0.1}$$

$$= 0.189 \quad (3dp)$$

6.10) $n = 100$, $r = 100$; WANT 90 % CL LL on p.

$$P(100;\,100,p) = p^{100}(1-p)^0 \frac{100!}{100!\,0!} = p^{100}$$

$$\therefore \text{Require} \int_{p_{LL}}^{1} P(100;\,100,p)\,dp = 0.9\int_0^1 P(100;\,100,p)\,dp$$

$$\therefore \int_{p_{LL}}^{1} p^{100}\,dp = 0.9\int_0^1 p^{100}\,dp$$

$$\therefore \frac{1}{101}\left[p^{101}\right]_{p_{LL}}^{1} = \frac{0.9}{101}\left[p^{101}\right]_0^1$$

$$\therefore 1 - p_{LL}^{101} = 0.9$$

$$\therefore p_{LL} = \sqrt[101]{0.1}$$

$$= 0.977 \quad (3dp)$$

6.11) PDF $= e^{-x} \times$ NORM let $A =$ NORM

$$0.9 = \frac{\int_0^{x_{UL}} \cancel{A} e^{-x} \cdot \cancel{P(x)}\,dx}{\int_0^{\infty} \cancel{A} e^{-x} \cdot \cancel{P(x)}\,dx}$$

A cancels

let $P(x) =$ constant $x \in [0,\infty]$

so prior also cancels.

$$\therefore 0.9 = \int_0^{x_{UL}} e^{-x}\,dx \Big/ \int_0^{\infty} e^{-x}\,dx$$

$$\therefore 0.9\left[-e^{-x}\right]_0^{\infty} = \left[-e^{-x}\right]_0^{x_{UL}}$$

$$\therefore 0.9 = 1 - e^{-x_{UL}} \Rightarrow e^{-x_{UL}} = 0.1$$

$$\ln 0.1 = -x_{UL}$$

$$\Rightarrow x_{UL} = 2.303$$

6.12) $r=0$ determine 95% CL UL on λ

$$P(r,\lambda) = \frac{\lambda^r e^{-\lambda}}{r!} = e^{-\lambda}$$

∴ solve

$$\int_0^{\lambda_{UL}} e^{-\lambda}\, d\lambda = 0.95 \int_0^{\infty} e^{-\lambda}\, d\lambda$$

$$\therefore \left[-e^{-\lambda}\right]_0^{\lambda_{UL}} = 0.95\left[-e^{-\lambda}\right]_0^{\infty}$$

$$\therefore 1-e^{-\lambda_{UL}} = 0.95$$

$$\therefore \lambda_{UL} = -\ln 0.05$$

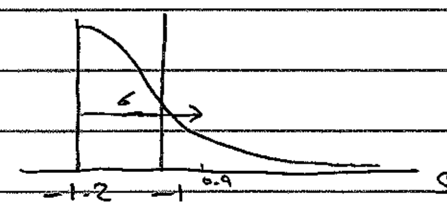
$$= 2.996$$

6.13) $S = -1.2 \pm 0.3$; assume errors are gaussian

S is physical for $S \in [-1,+1]$; use this to define prior:

$$P(S) = C \qquad S\in[-1,1]$$
$$= 0 \qquad \text{elsewhere}$$

$$\therefore 0.9 = \frac{\int_{-\infty}^{S_{UL}} P(S)\, G(S,-1.2,0.3)\, dS}{\int_{-\infty}^{\infty} P(S)\, G(S,-1.2,0.3)\, dS}$$



DENOMINATOR:

$$\text{NORM} = C\int_{-1}^{1} G(S,-1.2,0.3)\, dS$$

$$= C\int_{z=0.666}^{7.\dot{3}} G(z)\, dz \simeq C\int_{0.666}^{\infty} G(z)\, dz$$

$$= C\,(1-0.7454)$$

$$= 0.2546\, C$$

NUMERATOR:

$A = C\int_{-1}^{S_{UL}} G(S,-1.2,0.3)\, dS$ ∴ Solve for $A = 0.22914C$ to get S_{UL}.

6.13) contd. $\therefore \int_{0.666}^{z_{UL}} G(z)\, dz = 0.22914$

$$\int_{-\infty}^{0.666} G(z)\, dz \simeq 0.7454$$

WANT z_{UL} CORRESPONDING TO $\int_{-\infty}^{z_{UL}} G(z)\, dz = 0.97454$

$$\Rightarrow z_{UL} = 1.95$$
$$\Rightarrow S = -0.615$$

6.14) Given an expected background yield of one event, and an observed yield of one, determine the 90% CL upper limit on the signal yield μ.

$$P(N, \mu, B) = \frac{(\mu + B)^N e^{-(\mu+B)}}{N!}$$

$$\therefore P(1, \mu, B) = (\mu + 1)\, e^{-(\mu+1)}$$

$\therefore$ Want to solve for

$$\int_0^{\mu_{UL}} P(1,\mu,1)\, d\mu = 0.9 \int_0^{\infty} P(1,\mu,1)\, d\mu$$

$$\int_0^{\infty} P(1,\mu,1)\, d\mu = \int_0^{\infty} \mu e^{-(\mu+1)} + e^{-(\mu+1)}\, d\mu$$

$$= \left[-\mu e^{-(\mu+1)} - e^{-(\mu+1)}\right]_0^{\infty} + \int_0^{\infty} e^{-(\mu+1)}\, d\mu$$

$$= \left[-\mu e^{-(\mu+1)} - 2e^{-(\mu+1)}\right]_0^{\infty}$$

$$= \frac{2}{e}$$

$$\therefore \text{RHS} = \frac{1.8}{e}$$

6.14) contd.

$$\text{LHS} = \left[-\mu e^{-(\mu+1)} - 2e^{-(\mu+1)} \right]_0^{\mu_{UL}}$$

$$= \left[-\mu_{UL} e^{-(\mu_{UL}+1)} - 2e^{-(\mu_{UL}+1)} \right] + 2$$

$$\therefore \quad 2 - \frac{1.8}{e} = \mu_{UL} e^{-(\mu_{UL}+1)} + 2e^{-(\mu_{UL}+1)}$$

$$\frac{2e - 1.8}{e} = (\mu_{UL} + 2)\, e^{-(\mu_{UL}+1)}$$

This can be numerically solved to give $\mu_{UL} \leq 2.126$ 90 CL.

6.15) Use the unified approach to determine the 90% CL interval for N, given an expectation of two signal and two background events for a rare process.

$N_{sig} = 2$

$N_{bg} = 2$

$$P(N|N_{sig}) = \frac{(N_{sig}+N_{bg})^N \, e^{-(N_{sig}+N_{bg})}}{N!}$$

$$\mu_{BEST} = \text{MAX}\left[0, N - N_{bg}\right]$$

N	$P(N\|N_{sig})$	$P(N\|\mu_{BEST})$	R	RANK	CL (interval sum)
0	0.0183	0.1353	0.1353	9	0.9786
1	0.0733	0.2707	0.2707	7	0.9306
2	0.1465	0.2707	0.5413	5	0.7977
3	0.1954	0.2240	0.8720	3	0.5470
4	0.1954	0.1954	1	1	0.1954
5	0.1563	0.1755	0.8907	2	0.3517
6	0.1042	0.1606	0.6487	4	0.6512
7	0.0595	0.1490	0.3996	6	0.8573
8	0.0298	0.1396	0.2133	8	0.9603

INTERVAL $N \in [1, 7]$ has 93.06% CL

$\therefore$ the interval desired is $N \in [1, 7]$. This over covers, rather than a 90% C.L.; this is a 93.06% CL.