3.1) $\Omega = \{0.5, 0.9, 1.2, 1.5, 1.8, 2.0, 3.4, 4.1, 5.0, 5.1, 7.5, 8.5\}$

$K = \{0.7, 0.8, 1.1, 1.2, 1.5, 1.8, 1.9, 2.0, 2.5, 2.6, 2.9, 3.5\}$

$$\mu_x = \frac{1}{N} \sum_{i=1}^{N} x_i = \bar{x} = \langle x \rangle$$

$$\sigma_x^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2$$

$$\sigma_x = \sqrt{\sigma_x^2}$$

$$\gamma = \frac{1}{N \sigma_x^3} \sum_{i=1}^{N} (x_i - \bar{x})^3$$

$\langle \Omega \rangle = 3.458$	$\langle K \rangle = 1.875$
$\sigma_\Omega = 2.635$	$\sigma_K = 0.872$
$\sigma_\Omega^2 = 6.941$	$\sigma_K^2 = 0.760$
$\gamma_\Omega = 0.607$	$\gamma_K = 0.290$

3.2) $$\Sigma_{\Omega K} = \begin{pmatrix} \sigma_\Omega^2 & \sigma_{\Omega K} \\ \sigma_{\Omega K} & \sigma_K^2 \end{pmatrix}$$

$$\sigma_{\Omega K} = \frac{1}{N-1} \sum_i (\Omega_i - \bar{\Omega})(K_i - \bar{K})$$

$$= 2.236$$

$$\therefore \Sigma_{\Omega K} = \begin{pmatrix} 6.941 & \cancel{0.974}\ 2.236 \\ \cancel{0.974}\ 2.236 & 0.760 \end{pmatrix}$$

3.3) $$\rho_{\Omega K} = \frac{\sigma_{\Omega K}}{\sigma_\Omega \sigma_K} \Rightarrow \rho = \begin{pmatrix} 1 & 0.9734 \\ 0.9734 & 1 \end{pmatrix}$$

3.4) $V = \begin{pmatrix} 6.941 & 2.236 \\ 2.236 & 0.760 \end{pmatrix}$

$$\lambda_{\pm} = \frac{(\sigma_x^2 + \sigma_y^2) \pm \sqrt{(\sigma_x^2 + \sigma_y^2)^2 - 4(\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2)}}{2}$$

$\therefore \lambda_+ = 7.665$

$\lambda_- = 0.036$

$\psi_+ = \begin{pmatrix} 1 \\ 0.324 \end{pmatrix}$ or $\psi_+^{NORM} = \begin{pmatrix} 0.951 \\ 0.308 \end{pmatrix}$

$\psi_- = \begin{pmatrix} 1 \\ -3.088 \end{pmatrix}$ $\psi_-^{NORM} = \begin{pmatrix} 0.308 \\ -0.951 \end{pmatrix}$

$V' =$ diagonal error matrix $= \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$

$= \begin{pmatrix} 7.665 & 0 \\ 0 & 0.036 \end{pmatrix}$

which can be verified by combining eigenvectors into a matrix $P = (\psi_+ \psi_-)$ & then computing $P^{-1} V P = V'$.

3.5) $\Omega = \{1.0, 2.5, 3.0, 4.0, 4.5, 6.0\}$

$$\langle \Omega \rangle = \frac{1}{N} \sum_i \Omega_i = 3.5$$

$$\sigma_\Omega^2 = \frac{1}{N-1} \sum_i (\Omega_i - \bar{\Omega})^2 = 3$$

$\therefore \sigma_\Omega = 1.732$ (4 sf) $(= \sqrt{3})$ as $\sigma_\Omega = \sqrt{\text{Variance}(\Omega)}$

$$\gamma_\Omega = \frac{1}{n\sigma^3} \sum_{i=1}^{n} (\Omega_i - \bar{\Omega})^3 = 0$$

3.6) $\Omega = \{0.5, 1.0, 1.5, 1.6, 3.0, 2.1, 2.5\}$

formulae required – as above.

$\langle \Omega \rangle = 1.74$ (2dp)

$\sigma_\Omega^2 = 0.7429$ (4 dp)

$\sigma_\Omega = 0.8619$ (4dp)

$\gamma_\Omega = 0.0266$ (4dp)

3.7)

x	y	z
0.10	1.0	0.1
0.22	1.1	-0.2
0.25	1.1	0.3
0.50	1.2	0.4
0.55	1.3	0.1
0.70	1.4	-0.4
0.80	1.4	0.1
0.90	1.3	-0.1
1.00	1.6	0.6
1.11	1.5	0.7
1.12	1.4	-0.3

$\bar{x} = 0.659$

$\bar{y} = 1.300$

$\bar{z} = 0.118$

$\sigma_x = 0.364$

$\sigma_y = 0.184$

$\sigma_z = 0.357$

$$V = \begin{pmatrix} 0.132 & 0.061 & 0.019 \\ 0.061 & 0.034 & 0.017 \\ 0.019 & 0.017 & 0.128 \end{pmatrix}$$

$$\therefore \rho = \begin{pmatrix} 1 & 0.910 & 0.146 \\ 0.910 & 1 & 0.258 \\ 0.146 & 0.258 & 1 \end{pmatrix}$$

3.8)

x	y	z
0.0	0.9	-0.1
0.2	1.1	-0.2
0.3	1.2	0.1
0.4	1.2	0.2
0.5	1.3	0.1
0.7	1.4	0.0
0.8	1.5	0.2
0.9	1.3	0.1
1.0	1.6	0.5
0.9	1.5	0.6
1.1	1.3	0.3

$\bar{x} = 0.618$

$\bar{y} = 1.300$

$\bar{z} = 0.164$

$\sigma_x = 0.360$

$\sigma_y = 0.200$

$\sigma_z = 0.238$

$$V = \begin{pmatrix} 0.130 & 0.060 & 0.063 \\ 0.060 & 0.040 & 0.036 \\ 0.063 & 0.036 & 0.057 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1.000 & 0.833 & 0.733 \\ 0.833 & 1.000 & 0.757 \\ 0.733 & 0.757 & 1.000 \end{pmatrix}$$

3.9

x	0.5	0.7	0.8	0.9	1.1	1.3
y	0.9	0.8	1.1	1.2	1.2	1.0

$$\rho = 1 - \frac{6}{N(N^2-1)} \sum_{i=1}^{N} d_i^2$$

R_x	1	2	3	4	5	6
R_y	2	1	4	5.5	5.5	3
d^2	1	1	1	2.25	0.25	9

$\therefore \rho = 0.5857$ (4dp)

3.10) $$\rho = 1 - \frac{6}{N(N^2-1)} \sum_i d^2$$

x	0.1	0.3	0.2	0.0	0.4	0.5	0.1	0.2	0.6	0.5
y	0.5	0.7	0.2	0.3	0.8	0.1	0.9	0.0	0.4	0.6
R_x	2.5	6	4.5	1	7	8.5	2.5	4.5	10	8.5
R_y	6	8	3	4	9	2	10	1	5	7
d^2	$12\frac{1}{4}$	4	$2\frac{1}{4}$	9	4	$42\frac{1}{4}$	$56\frac{1}{4}$	$12\frac{1}{4}$	25	$2\frac{1}{4}$

$\therefore \rho = -0.0136$ (4 dp).

3.11) $$\begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} = U^T \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_v^2 \end{pmatrix} U = V$$

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \det U = 1, \quad U^T = U^{-1}$$

$$\therefore V = \begin{pmatrix} \sigma_u^2 \cos^2\theta + \sigma_v^2 \sin^2\theta & \sigma_u^2 \sin\theta\cos\theta - \sigma_v^2 \sin\theta\cos\theta \\ \sigma_u^2 \cos\theta\sin\theta - \sigma_v^2 \cos\theta\sin\theta & \sigma_u^2 \sin^2\theta + \sigma_v^2 \cos^2\theta \end{pmatrix}$$

$$\therefore \sigma_{xy} = (\sigma_u^2 - \sigma_v^2)\cos\theta\sin\theta \quad \text{--- (1)}$$
$$\sigma_x^2 = \sigma_u^2 \cos^2\theta + \sigma_v^2 \sin^2\theta \quad \text{--- (2)}$$
$$\sigma_y^2 = \sigma_u^2 \sin^2\theta + \sigma_v^2 \cos^2\theta \quad \text{--- (3)}$$

From 2: $$\sigma_u^2 = \frac{\sigma_x^2 - \sigma_v^2 \sin^2\theta}{\cos^2\theta}$$

Sub → (3):

$$\sigma_y^2 = \frac{\sigma_x^2 - \sigma_v^2 \sin^2\theta}{\cos^2\theta} \sin^2\theta + \sigma_v^2 \cos^2\theta$$

$$\therefore \sigma_y^2 \cos^2\theta - \sigma_x^2 \sin^2\theta = \sigma_v^2(\cos^2\theta - \sin^2\theta)$$

$$\therefore \boxed{\sigma_v^2 = \frac{\sigma_y^2 \cos^2\theta - \sigma_x^2 \sin^2\theta}{\cos^2\theta - \sin^2\theta}}$$

contd.

3.11) contd.

$$\therefore \sigma_y^2 = \sigma_u^2 \sin^2\theta + \cos^2\theta \left[\frac{\sigma_y^2 \cos^2\theta - \sigma_x^2 \sin^2\theta}{\cos^2\theta - \sin^2\theta} \right]$$

$$\therefore \frac{\sigma_y^2 \cos^2\theta - \sigma_y^2 \sin^2\theta - \sigma_y^2 \cos^4\theta + \sigma_x^2 \sin^2\theta \cos^2\theta}{\cos^2\theta - \sin^2\theta} = \sigma_u^2 \sin^2\theta$$

$$\therefore \sigma_u^2 = \frac{\sigma_y^2 (\cos^2\theta - 1) + \sigma_x^2 \cos^2\theta}{\cos^2\theta - \sin^2\theta}$$

$$\therefore \boxed{\sigma_u^2 = \frac{\sigma_x^2 \cos^2\theta - \sigma_y^2 \sin^2\theta}{\cos^2\theta - \sin^2\theta}}$$

Given $\sigma_{xy} = \left(\sigma_u^2 - \sigma_v^2\right) \cos\theta \sin\theta$

$$\therefore \sigma_{xy} (\cos^2\theta - \sin^2\theta) = \left(\sigma_x^2 - \sigma_y^2\right) \sin\theta \cos\theta$$

$$\therefore \sigma_{xy} \cos 2\theta = \frac{1}{2} \left(\sigma_x^2 - \sigma_y^2\right) \sin 2\theta$$

$$\therefore \tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2}$$

$$\Rightarrow \boxed{\theta = \frac{1}{2} \arctan \left[\frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2} \right]}$$

3.12) Diagonalise $V = \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix}$

$$\lambda_\pm = \frac{1}{2}\left[(\sigma_x^2 + \sigma_y^2) \pm \sqrt{(\sigma_x^2 + \sigma_y^2)^2 - 4(\sigma_x^2\sigma_y^2 - \sigma_{xy}^2)}\right]$$

$\lambda_+ = 1.2$

$\lambda_- = 0.8$

$$\therefore V' = \begin{pmatrix} 1.2 & 0 \\ 0 & 0.8 \end{pmatrix}$$

3.13) Diagonalise $V = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix}$

$\lambda_+ = 2.207$

$\lambda_- = 0.793$

$$\therefore V' = \begin{pmatrix} 2.207 & 0 \\ 0 & 0.793 \end{pmatrix}$$

3.14) $\sigma_x = 1$, $\sigma_y = 0.5$ $\sigma_{xy} = 0.25$

$$\therefore V = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 0.25 \end{pmatrix} = U^T V' U$$

$$\theta = \frac{1}{2}\arctan\left[\frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2}\right] \Rightarrow \theta = 0.294 \text{ rad } (16.8°)$$

$$U = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix} = \begin{pmatrix} 0.957 & 0.290 \\ -0.290 & 0.957 \end{pmatrix}$$

3.15) $\sigma_x = 2$; $\sigma_y = 1.5$; $\sigma_{xy} = 1.0$

$\Rightarrow \theta = 0.426$ rad $(24.4°)$

$$\therefore U = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix} = \begin{pmatrix} 0.911 & 0.413 \\ -0.413 & 0.911 \end{pmatrix}$$