SDA 2 — PROBABILITY

TUTORIAL QUESTIONS

2.1) There are 4 aces in a deck of 52 cards

$\therefore$ P (ace) = 4/52 = 0.0769 (4dp)

2.2) TWO ACES DELT, ONE AFTER THE OTHER:

1^{st} draw P(ace1) = 4/52 = 0.0769 (4dp)

2^{nd} draw P(ace 2) = 3/51 = 0.0588 (4dp)

$\therefore P(\text{ace1}) \times P(\text{ace2}) = \dfrac{4 \times 3}{52 \times 51} = 0.0045$ (4 dp)

2.3) NUMBER OF ACES OR 10 POINT CARDS IN A DECK: 20 (A, K, Q, J, 10)

OUT OF 52 CARDS.

$\therefore P(A \mid K \mid Q \mid J \mid 10) = \frac{20}{52} = 0.3846$ (4dp)

2.4) GAME = BLACK JACK $N_{PLAYERS} = 2$

WHAT IS P_r OF BEING DELT ACE, FOLLOWED BY K/Q/J/10 ?

3 EVENTS: EVT 1: DELT ACE

EVT 2: PLAYER 2 DELT SOMETHING

EVT 3: DELT K/Q/J/10.

P = P(EVT 1) × P(EVT 2) × P(EVT 3)

P(EVT1) = $\frac{4}{52}$ = 0.0769 (4 dp) (FROM Q 2.1)

P(EVT2) $\longrightarrow$ 2^{nd} PLAYER GETS 10 pt CARD (A) P = 16/51

2^{nd} PLAYER GETS SOMETHING ELSE (B) P = 35/51

P(EVT 3 | EVT2-A) = 15/50

P(EVT3 | EVT2-B) = 16/50

SEQUENCE A: ACE : 10 pt : 10 pt $P_r = \frac{4}{52} \times \frac{16}{51} \times \frac{15}{50} = 0.0072$ (4dp)

SEQUENCE B: ACE : NOT 10 pt : 10 pt $P_r = \frac{4}{52} \times \frac{35}{51} \times \frac{16}{50} = 0.0169$ (4dp)

$\therefore$ TOTAL PROB = 0.0072 + 0.0169

= 0.0241 (4dp)

2.5) $P(\text{rain} \mid \text{rain forecast}) = 0.7$

$P(\text{rain} \mid \text{not forecast}) = 0.05$

$P(\text{rain}) = 30/365 = 0.0822$

$P(\text{no rain}) = 1 - P(\text{rain}) = 0.9178$

Bayes Theorem gives:

$$P(\text{rain forecast} \mid \text{rain}) = \frac{P(\text{rain} \mid \text{rain forecast})\ P(\text{rain})}{P(\text{data})}$$

$P(\text{data}) = P(\text{rain} \mid \text{rain forecast})\ P(\text{rain}) + P(\text{rain} \mid \text{not forecast})\ P(\text{no rain})$

$= 0.1034$ (4dp)

$\therefore P(\text{rain forecast} \mid \text{rain}) = 0.5563$ (4dp)

$\Rightarrow$ The probability that it will rain tomorrow is 55.6%.

2.6) $P(\text{storm} \mid \text{storm forecast}) = 0.95$

$P(\text{storm} \mid \text{not forecast}) = 0.02$

$P(\text{storm}) = 20/365 = 0.0548$ (4dp)

$P(\text{no storm}) = 1 - P(\text{storm}) = 0.9452$ (4dp)

Bayes Theorem gives:

$$P(\text{storm forecast} \mid \text{storm}) = \frac{P(\text{storm} \mid \text{storm forecast})\ P(\text{storm})}{P(\text{data})}$$

where $P(\text{data}) = P(\text{storm} \mid \text{storm forecast})\ P(\text{storm}) + P(\text{storm} \mid \text{not forecast})\ P(\text{no storm})$

$= 0.0710$ (4dp)

$\therefore P(\text{storm forecast} \mid \text{storm}) = 0.7336$ (4dp)

$\Rightarrow$ The probability that it will be stormy ~~storm~~ tomorrow is 73.4%.

2.7) $P(\text{rain} \mid \text{rain forecast}) = 0.7$

$P(\text{rain} \mid \text{not forecast}) = 0.05$

$P(\text{rain}) = 100/365 = 0.2740$ (4dp)

$P(\text{no rain}) = 1 - P(\text{rain}) = 0.7260$ (4dp)

BAYES THEOREM GIVES:

$$P(\text{rain forecast} \mid \text{rain}) = \frac{P(\text{rain} \mid \text{rain forecast})\, P(\text{rain})}{P(\text{data})}$$

$P(\text{data}) = P(\text{rain} \mid \text{rain forecast})\, P(\text{rain}) + P(\text{rain} \mid \text{not forecast})\, P(\text{no rain})$

$= 0.2281$ (4dp)

$\therefore P(\text{rain forecast} \mid \text{rain}) = 0.841$ (3dp) i.e. 84% chance of rain.

If one assumes $P(\text{rain}) = P(\text{no rain}) = 0.5$ then we obtain

$P(\text{rain forecast} \mid \text{rain}) = 0.933$ i.e. 93% chance of rain.

2.8) $P(\text{rain} \mid \text{rain forecast}) = 0.90$

$P(\text{rain} \mid \text{not forecast}) = 0.05$

$P(\text{rain}) = 30/365 = 0.0822$ (4dp)

$P(\text{no rain}) = 1 - P(\text{rain}) = 0.9178$ (4dp)

As above one can use Bayes theorem to obtain:

$P(\text{data}) = 0.1199$ 4dp

Hence,

$P(\text{rain forecast} \mid \text{rain}) = 0.6171$ (4dp).

If one were to choose $P(\text{rain}) = P(\text{no rain}) = 0.5$, the corresponding result would be a 94.7 % chance of rain.

2.9) $P(\text{empty room}) = 0.5$

$P(\text{staircase}) = 0.5$

Using Bayes theorem : $P(\text{data}) = \sum_{i=1}^{2} P(\text{data} | \text{hypothesis}) P(\text{hypothesis})$

$= 0.5^2 + 0.5^2$

$= 0.5$

staircase or empty room

& $P(\text{finding empty room} \mid \text{open one door}) = \dfrac{0.5 \times 0.5}{0.5} = 0.5$

2.10) MONTY HALL PROBLEM

SOLUTION: CHANGE DOORS TO MAXIMISE PROBABILITY OF WINNING

$P_s = 2/3$.

TWO PARTS TO THE PROBLEM

A: SELECT 1 OF 3 DOORS

B: RE EVALUATE POSITION.

STEP A

PRIORS: $\forall$ DOORS ARE EQUAL $P(A) = \frac{1}{3}$

$P(B) = \frac{1}{3}$

$P(C) = \frac{1}{3}$

$\therefore P(\text{data}) = \sum_{i=1}^{3} \underbrace{P(\text{data} \mid \text{door}_i)}_{1/3} \underbrace{P(\text{door}_i)}_{1/3}$

$\Rightarrow P(\text{data}) = \sum_{i=1}^{3} \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}$

STEP B: NOW WE HAVE TWO DOORS ; STICK OR SWITCH ?

$P(\text{STICK}) = P(\text{SWITCH}) = 1/2$

$P(\text{STICK} \mid \text{WIN CAR}) = \dfrac{1/2 \times 1/2}{P(\text{data})}$

$P(\text{data}) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = 1/2$

$\therefore P(\text{STICK} \mid \text{WIN CAR}) = 1/2$

$\therefore$ Probability of winning a car picking one door & sticking $= 1/3 \times 1/2 = 1/6$; 2 combinations of this happening so total prob $= 1/3$.

$\Rightarrow$ switch choice to have a 2/3 chance of winning

2.11) $y = N(3-x)^2 \qquad x \in [-3, 3]$

For y to be a PDF $\int_{-3}^{3} y\, dx = 1$

$\therefore N \int_{-3}^{3} (3-x)^2 = 1 \qquad = N\int_{-3}^{3} 9 - 6x + x^2 \, dx$

$= N\left[9x - 3x^2 + \frac{x^3}{3}\right]_{-3}^{3}$

$= 72N$

$\therefore N = 1/72$

$h = \frac{(3-x)^2}{9}$ as $h_{max} = 1$ when $x = 0$

2.12) $y = N(1 + x + x^2) \qquad x \in [0, 2]$

$N \int_{0}^{2} 1 + x + x^2 \, dx = 1$

$= N\left[x + \frac{x^2}{2} + \frac{x^3}{3}\right]_0^2$

$= 6\,^2/_3$

$= 20/3$

$\therefore N = 3/20$

h_{max} is at $x = 2 \quad \therefore \quad h = \frac{1}{7}(1 + x + x^2)$

2.13) $y = N e^{-t/\tau}$

$\tau = 2.2\,\mu s$

$t \in [0, 20]\,\mu s$

$N \int_0^{20} e^{-t/\tau}\, dt = 1$

$= -N.\ \tau \left[e^{-t/\tau}\right]_0^{20}$

$= N\left\{-2.2\left(e^{-20/2.2} - e^0\right)\right\}$

$= -2.2N\,(0.00011 - 1)$

$= 2.19975\,N$

2.13) CONTD

$$\therefore N = \frac{1}{2.19975} = 0.4546 \quad (4dp)$$

$$\therefore P = 0.4546\, e^{-t/2.2}$$

As L_{MAX} occurs for $t=0$;

$$L = e^{-t/2.2}$$

2.14) Given $L = e^{-x}$

$$L(x=1) = e^{-1} = 1/e$$

$$L(x=2) = e^{-2} = 1/e^2$$

$$\therefore R = \frac{L(x=1)}{L(x=2)} = e \quad (\text{approx } 2.7182)$$

2.15) $L = e^{-x^2}$

$$L(x=1) = e^{-1}$$

$$L(x=-1) = e^{-1}$$

$$\therefore R = \frac{L(x=-1)}{L(x=1)} = 1$$