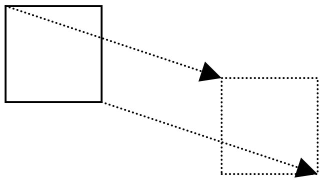
Dynamics of Rigid Bodies

A *rigid body* is one in which the distances between constituent particles is constant throughout the motion of the body, i.e. it keeps its shape.

There are two kinds of rigid body motion:

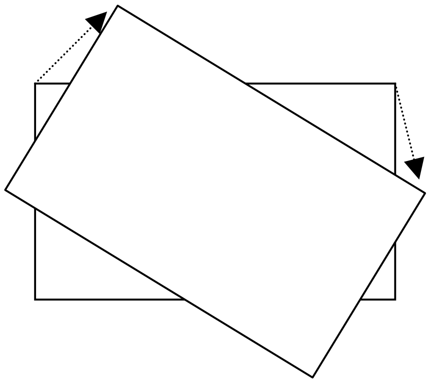
1. Translational



Rectilinear forces acting.

Particles move on straight-line paths.

2. Rotational



Rotational forces, i.e. torques, acting.

Particles move on circular paths

Kinematics of Rigid Bodies

$s = R\theta \qquad \omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Angle θ is in radians

Let α be a constant.

Then, $\frac{d^2\theta}{dt^2} = \alpha$ Integrating, $\frac{d\theta}{dt} = \alpha t + C$

Let $\omega = \omega_0$ at t = 0, $\omega = \frac{d\theta}{dt} = \alpha t + \omega_0$

Integrating, $\theta = ½\alpha t^2 + \omega_0 t + c$

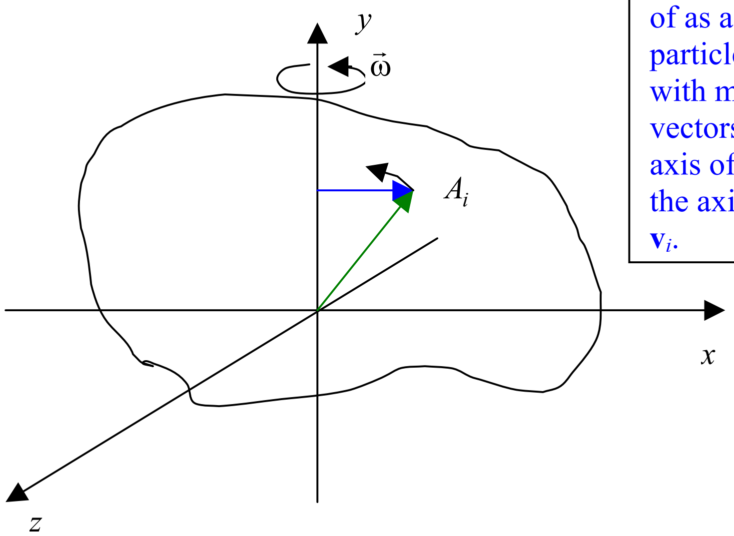
Let $\theta = 0$ at t = 0, $\boxed{\theta = ½\alpha t^2 + \omega_0 t}$

COMPARE THESE WITH THE EQUATIONS FOR LINEAR MOTION

Note: θ is in radians

Angular Momentum

Angular momentum is the rotational equivalent of linear momentum. It is a conserved quantity



A rigid body can be thought of as a large number of particles i, at positions A_i, with masses m_i, at position vectors from an origin on the axis of $\mathbf{r}_i$, at distances R_i from the axis, and with velocities $\mathbf{v}_i$.

The particle at the point A_i has linear momentum $\mathbf{p} = m_i\mathbf{v}_i$ and *Angular Momentum* about the origin

$$\begin{aligned}\mathbf{L}_i &= \mathbf{r}_i \times \mathbf{p}_i \\ &= m_i \mathbf{r}_i \times \mathbf{v}_i = m_i r_i v_i \hat{\mathbf{n}}\end{aligned}$$

where $\hat{\mathbf{n}}$ is unit vector normal to both $\mathbf{r}_i$ and $\mathbf{p}_i$.

Note that **L** is not generally parallel to the axis of rotation.

Moment of Inertia

The component of the angular momentum in the y-direction, i.e. along th axis of rotation, is

$$L_{iy} = m_i r_i v_i \sin\theta_i$$

However, $r_i \sin\theta_i = R_i$, so

$$L_i = m_i R_i v_i$$

The magnitude of ω_i is the same for all points, so we drop the index i, and $v_i = R_i\omega$. Then

$$L_{iy} = m_i R_i^2 \omega$$

Summing or integrating over all points,

$$L_y = L_{1y} + L_{2y} + \ldots = \sum_i L_{iy} = \omega \sum_i m_i R_i^2$$

This sum, $I = \sum_i m_i R_i^2$ is called the ***Moment of Inertia***, and $\underline{\underline{L = I\omega}}$ (c.f. $p = mv$)

Moment of Inertia of Potato-Shapes

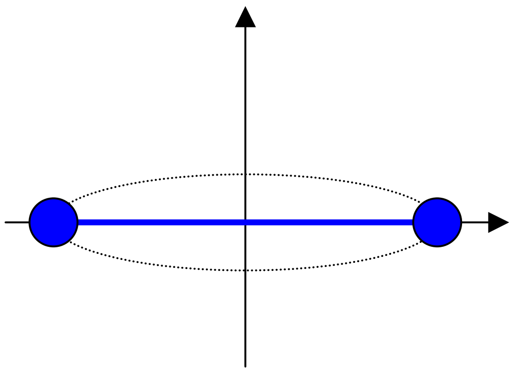
The Moment of Inertia depends on the axis of rotation.

The Angular Momentum is generally not parallel to the axis of rotation.

For a body of general shape (an asteroid, a potato . . .) there are three mutually perpendicular axes for which the angular momentum *is parallel* to the axis. These are called the ***Principal Axes*** of intertia and the moments of inertia about them are the ***Principal Moments of Inertia.***

For bodies of higher symmetry than potatoes, the Principal Axes are generally Axes of Symmetry.

Angular Momentum Examples:



Two masses m are going round the z-axis, at radius R, in the x-y plane, with speed $v = R\omega$.

$\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ *for each mass*

$|\mathbf{r}| = R$

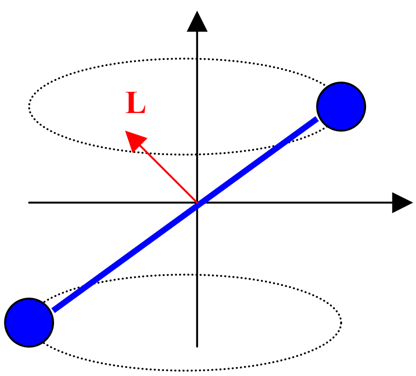
$|\mathbf{v}| = R\omega$

$|\mathbf{L}| = 2mR^2\omega$ *for the pair*

Direction of **L** is at right angles to **r** and **v**, i.e. in same direction as ω, i.e.

$$\mathbf{L} = 2mR^2\boldsymbol{\omega} = I\boldsymbol{\omega}$$

and $I = 2mR$



Now incline the masses to the axis.

$\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ *for each mass*

$|\mathbf{r}| = R$

$|\mathbf{v}| = R\omega \sin\varphi$

$|\mathbf{L}| = 2mR^2\omega \sin\varphi$ *for the pair*

Direction of **L** is at right angles to **r** and **v**, i.e. at φ to **ω**, i.e. it *precesses* about z.

$L_z = 2mR^2\sin^2\varphi\ \omega$, and $I = 2mR^2\sin^2\varphi$ about the z-axis.

Calculation of Moments of Inertia

$$I = \sum_i m_i R_i^2$$

R_i is distance to axis of rotation

If an object is considered to consist of elemental particles of mass *dm*, then

$$dm = \rho dV$$

and the sum becomes an integral over the volume. If the density ρ is constant, it comes out of the integral and

$$I = \int_V R^2 dm = \rho \int_V R^2 dV$$

Note that $\int_V R^2 dV$ is a purely *geometrical factor*

Example: Moment of Inertia of a thin rod rotated about one end.

Cross-sectional area S, length L, density ρ.

Element dV is disc of area S, distasnce from axis x, thickness dx.

$$\left(i.e.\ dV = Sdx,\ dm = \rho Sdx,\ dI = \rho Sx^2 dx \right)$$

So we have

$$I = \rho S \int_{x=0}^{L} x^2 dx = \frac{1}{3} \rho S L^3 = \frac{1}{3} M L^2$$

Example: Moment of Inertia of the same thin rod rotated about its centre.

$$I = \rho S \int_{x=-L/2}^{L/2} x^2 dx = \frac{1}{3} \rho S \left(\frac{L^3}{8} - -\frac{L^3}{8} \right) = \frac{1}{12} M L^2$$

In general, $I = Mk^2$. The length k is a characteristic length of the object, called the *Radius of Gyration* – compare with Centre of Gravity.

Parallel Axis Theorem

Consider an object rotating around its Centre of Mass, with the *z*-axis as the axis of rotation. We know the moment of inertia

$$I_{CM} = \sum_i m_i R_i^2 = \sum_i m_i \left(x_i^2 + y_i^2\right).$$

What would be the moment of inertia about the axis *P*, parallel to the *z*-axis but offset at $x = a$, $y = b$?

$$I_P = \sum_i m_i \left[(x_i - a)^2 + (y_i - b)^2\right]$$

$$= \sum_i m_i \left[\left(x_i^2 + y_i^2\right) - 2ax_i - 2by_i + \left(a^2 + b^2\right)\right]$$

Recall that the definition of the Centre of Mass gives $x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i}$. Since we have placed the Centre of mass at the origin, our term in x_i vanishes. Similarly for y_i. So,

$$I_P = \sum_i m_i \left(x_i^2 + y_i^2\right) + \left(a^2 + b^2\right) \sum_i m_i$$

$$= I_{CM} + Md^2$$

Compare this result with the thin rod in the earlier example. $I_{end} = I_{CM} + Md^2$, and $d = ½L$.

$$I_{CM} = I_{end} - \frac{ML^2}{4} = \frac{ML^2}{3} - \frac{ML^2}{4} = \frac{ML^2}{12}$$

which is what we found by direct calculation.

Torque

To cause a body to rotate we need to apply forces which do not pass through the axis of rotation. The effect of such a force depends on its magnitude and direction and also on how far from the axis it is.

In vector terms, this is called the *Torque*. Its scalar magnistude is often called the *Moment*. It is given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

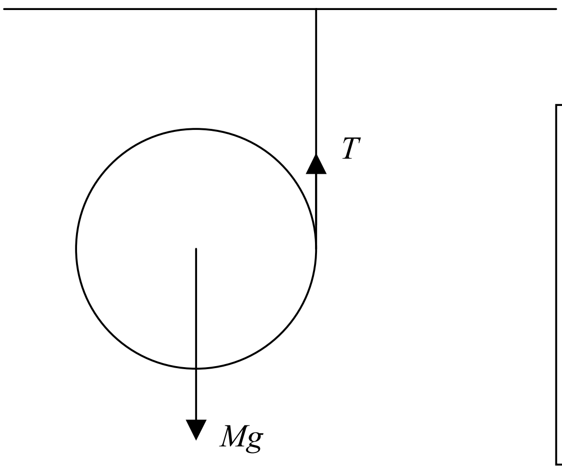
$$|\vec{\tau}| = RF$$

Torque is the rotational analogue of force in linear mechanics. As force is rate of change of linear momentum, so torque is rate of change of angular momentum,

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt} I\vec{\omega} = I\frac{d}{dt}\vec{\omega} = I\frac{d^2}{dt^2}\vec{\theta} = I\vec{\alpha}$$

$\vec{\tau} = I\vec{\alpha}$ is to be compared with $\vec{F} = m\vec{a}$

Example: The Yo-Yo



Vertical acceleration a
Angular acceleration α
$\tau = I\alpha$
$I = \frac{1}{2} MR^2$ *
$T = \frac{1}{2} MR\alpha$

* How would you calculate this?

Note that $a = \alpha R$ and that $Mg - T = Ma = MR\alpha$

So

$$Mg - \tfrac{1}{2}MR\alpha = MR\alpha$$

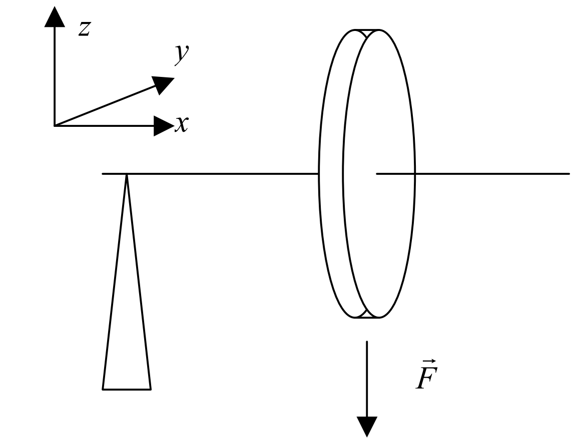
$$Mg - \tfrac{1}{2}Ma = Ma$$

$$g = \tfrac{3}{2}a$$

$$a = \tfrac{2}{3}g = 6.53\ \mathrm{m\,s^{-1}}$$

Note that $a < g$ and is independent of R and M.

Gyroscopes and Precession



Consider a gyroscope which is *not spinning*. It is released in the position shown. Its centre of mass is at $\vec{r}$.

Initially $\vec{L}$ is zero

Then $\vec{\tau} = \dfrac{d\vec{L}}{dt}$

$$d\vec{L} = \vec{\tau} dt$$

The torque causes the gyroscope to start rotating about the pivot, gaining angular velocity in the y-direction.

Now let the gyroscope be spinning initially. Its angular momentum $\vec{L}$ is not zero but lies along the axis of rotation, along the x-direction. Now, after a time dt,

$$\vec{L} \to \vec{L} + \vec{\tau} dt$$

which is still in the x-y plane, which has the same magnitude, but has changed direction by the amount

$$d\varphi = \frac{|\vec{\tau}| dt}{|\vec{L}|}$$

$$\Omega = \frac{d\varphi}{dt} = \frac{|\vec{\tau}|}{|\vec{L}|} = \frac{mgr}{I\omega}$$

This gives us an expression for the well-known rotation in the horizontal plane of a gyroscope, its PRECESSION.

For a simple disc, the moment of inertia $I = ½\, mR^2$ and $\Omega = \dfrac{gr}{½R^2\omega}$

Astronomical Example: The Earth is spinning and so acts as a gyroscope. Its axis of rotation makes an angle of 23º27′ to the normal to the plane of its orbit. There is a torque due to (tidal) gravitational effects of the Sun and Moon. As a consequence, the direction of the Earth's axis precesses (the Precession of the Equinoxes) with a period of 27725 years.

Consequently, while Isambard Brunel built the Box Tunnel (between Bath and Bristol on the Great Western Railway) so that the sun would shine through it at sunrise on his birthday, that condition will not be met for very long.

Kinetic Energy of Rotation.

For a body considered as a large number of point masses,

$$E_K = \tfrac{1}{2}\sum_i m_i v_i^2$$

$$= \tfrac{1}{2}\sum_i m_i R_i^2 \omega^2 = \tfrac{1}{2}\left(\sum_i m_i R_i^2\right)\omega^2$$

$$= \tfrac{1}{2} I\omega^2$$

Compare $E = \tfrac{1}{2}mv^2$

In the general case of linear translation motion together with rotation about the centre of mass,

$$E_K = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

Example: A ring rolls* down an inclined plane, height h_0, angle φ

* *Definition of Rolling:* Friction prevents slippage. No work is done against friction as the bottom of the ring touching the ramp is instantaneously at rest.

If the ring starts at height h_0 at zero linear and angular velocity, and rolls down to a height h, then at h,

$$\tfrac{1}{2} E_K = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = mg(h_0 - h)$$

Now, I for a ring of mass m, radius R is $I = mR^2$, and $v = R\omega$, so

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}mR^2\left(\frac{v}{R}\right)^2 = mg(h_0 - h)$$

$$v^2 = g(h_0 - h)$$

Compare frictionless sliding:

$$E_K = \tfrac{1}{2}mv^2 = mg(h_0 - h)$$

$$v^2 = 2g(h_0 - h)$$

The initial Potential Energy is used to supply both the rotational and translational kinetic energy, so the rolling motion slows down the tranlational.

Comparison of Dynamical Quantities in Translation and Rotation.

Translational		Rotational	
Position	$\vec{s}$	Angle	$\vec{\theta}$
Velocity	$\vec{v} = \frac{d\vec{s}}{dt}$	Angular velocity	$\omega = \frac{d\vec{\theta}}{dt}$
Acceleration	$\vec{a} = \frac{d^2\vec{s}}{dt^2}$	Angular acceleration	$\vec{\alpha} = \frac{d^2\vec{\theta}}{dt^2}$
Mass	m	Moment of Inertia	I
Momentum	$\vec{p} = m\vec{v}$	Angular Momentum	$\vec{L} = I\vec{\omega}$
Kinetic Energy	$E = ½mv^2 = \frac{p^2}{2m}$	. . . . .	$E = ½I\omega^2 = \frac{L^2}{2I}$
Force	$\vec{F} = \frac{d\vec{p}}{dt}$	Torque	$\vec{\tau} = \vec{F} \times \vec{r} = \frac{d\vec{L}}{dt}$
Power	$P = \vec{F}.\vec{v}$	. . . . .	$P = \vec{\tau}.\vec{\omega}$
Second Law	$\vec{F} = m\vec{a}$	. . . . .	$\vec{\alpha} = I\,\vec{\tau}$